**ECE 3100** **Final Examination** **Fall 2014**

**Wednesday, 03 December 2014**

**NAME:** ________________________________________ **netID:** _____

- Exam duration: 7:00 - 9:30 PM, that's 2 hours 30 minutes
- There are **11** questions to this exam
- Exams are collected at 9:30 PM, no extensions
- Write your answers on this set of sheets
- Write your calculations/reasonings *clearly*: no points awarded for what you *intended to write* or *meant*
- Write ALL calculations
- Everywhere, the vertical separator bar (as in, for example, $(X \mid Y)$) means 'given that' ($X$ given $Y$), and it also means '$X$ conditioned on $Y$'.
- You can use your calculator for simple calculations; NO programming functions, NO complex calculations (integrals, probability functions, ...) with calculators are allowed nor will count for points if used to that end.
- Writing a formula as an answer does NOT count for points (e.g., if you're asked to calculate the expectation value and you write for your answer $\mathrm{E}[X] = \sum_x x p_X(x)$, this is not an answer and counts for no points).
- Be careful of the values given in the problems, and round numbers appropriately.
- Circle your final answer
- There is Useful Stuff at the end of the exam (look at it NOW).
- Keep your cheat sheets after the exam
- If you are caught plagiarizing during the exam, you will get a grade of 0 for the exam.

**1. 14 Points (2 Points each). Answer each question below with True or False, no partial credit.**

(a) For discrete random variables and in certain cases, the sum of the probabilities of all of the sample points in the sample space can be less than 1.

(b) Suppose that var$(X+Y)$ =vax$(X)$+var$(Y)$. Then, $X$ and $Y$ are uncorrelated.

(c) If $Z$ is the Standard Normal random variable, then:

$$P(|Z| \geq \gamma) \leq \frac{1}{\gamma^2},$$

for all $\gamma > 0$.

(d) If $Z$ is a Standard Normal random variable, then $E[Z^3] = 0$.

(e) There exists at least one pair of random variables $X$ and $Y$ with $E[X^4] = 4$, $E[Y^4] = 1$, and $E[X^2Y^2] = 3$.

(f) If $X$ is a Bernouilli(1/2), then $E[X^n]$=1/2 for all $n$.

(g) Carl was awesome.

**2. 10 Points.** Consider a binary string of $n$ bits (*e.g.*, 1011000 has 7 bits). We call a *transition* whenever a given bit is different from the bit that precedes it. So for example, in the string 1011000, there are 3 transitions: from the first 1 to the 0 after it, and from that 0 to the 1 after it, and from the third 1 to the 0 after it. In digital circuits, transitions matter because they imply that more power is being consumed.

(a) **2 Points.** What is the maximum number of transitions that a binary string of length $n$ can have?

(b) **2 Points.** How many binary strings have the maximum number of transitions?

(c) **3 Points.** How many binary strings of length $n$ bits exactly $k$ transitions if $k \leq n-1$?

(d) **3 Points.** What is the total number of binary strings of length $n$?

**3. 5 Points.** A certain car insurance company, in an attempt to increase its revenues, looks at all of its clients' files and obtains the following information:

- All of their clients insure at least one car.
- 70% of their clients insure more than one car.
- 20% of their clients insure a luxury car.
- Of those clients that insure more than one car, 15% insure a luxury car.

Calculate the probability that, if one of their clients is chose at random, this client insures exactly one car and that this car is not a luxury car.

**4. 5 Points.** An electronic gizmo is made up of two circuits. The second circuit is only a backup for the first circuit: after turning the gizmo on at time 0, the second circuit is only used when the first circuit fails. The electronic gizmo itself fails entirely if both circuits fail. Let $X$ and $Y$ be the times at which the first and second circuits fail after turning the gizmo on, respectively. $X$ and $Y$ have a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-x}e^{-2y}, & \text{for } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the expected time a which the electronic gizmo fails (give a numerical answer).

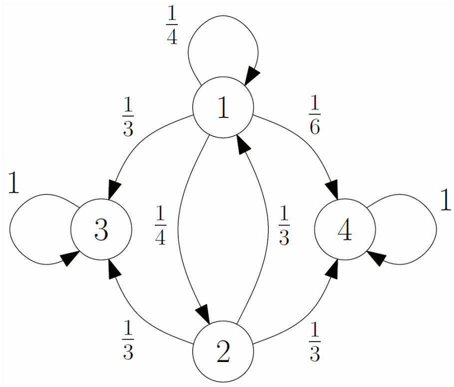
**5. 9 Points.** Two friendly alien races, one race with blue antennas and the other race with yellow antennas, are about to start arriving (independently of one another) on Earth. The aliens with blue and yellow antennas will be arriving according to Poisson process distributions with parameters $\lambda_b$ and $\lambda_y$, respectively. Humanity accepts them with open arms but the aliens will need to register with the newly formed Alien Registry Organization (ARO) when they begin to arrive. Let $t_1$ be the time that the ARO will be in operation until the first alien (of any antenna color) is registered. Let $B$ be the event that the first alien to be registered is one with blue antennas. Let $t_2$ be the time that the ARO will be in operation until at least one alien of both races is registered.

(a) **3 Points.** Calculate $\mu_1 = E[t_1]$ in terms of $\lambda_b$ and $\lambda_y$.

(b) **3 Points.** Calculate $p = P(B)$ in terms of $\lambda_b$ and $\lambda_y$.

(c) **3 Points.** Calculate $\mu_2 = E[t_2]$ in terms of $\lambda_b$ and $\lambda_y$.

**6. 15 Points.** Consider a Markov chain $X_n$ whose one-step transition probabilities are shown in the figure below.




(a) **3 Points.** What are the recurrent states?

(b) **3 Points.** Calculate $P(X_2 = 4 \mid X_0 = 2)$.

(c) **3 Points.** Suppose that you are given the values of $r_{ij}(n) = P(X_n = j \mid X_0 = i)$. Give a formula for $r_{11}(n+1)$ in terms of the $r_{ij}(n)$.

(d) **3 Points.** Find the steady-state probabilities $\pi_j = \lim_{n\to\infty} P(X_n = j \mid X_0 = i)$, or explain why they do not exist if they do not exist.

(e) **3 Points.** What is the probability of eventually visiting state 4, given that the initial state is $X_0 = 1$?

**7. 10 Points.** 25 plastic planks are laid out end-to-end to form a path that measures approximately 1000 meters. The plastic planks are made in molds. Since every plank is not exactly the same length, any variation in the length of any plank is entirely due to variations between the different molds. The length of a particular mold is modeled with a random variable $X$, and this length is independent of the length of all other molds. $X$ is uniformly distributed between $40-\sqrt{3}$ and $40+\sqrt{3}$ meters. Assume that a plank fabricated in a given mold has exactly the same length as the mold.

(a) **2 Points.** Calculate the mean and variance of $X$.

(b) **3 Points.** What is the probability that the resulting path will be within $1000 \pm 7.5$ meters if the 25 planks that are used are all made from the same mold?

(c) **3 Points.** What is the probability that the resulting path will be within $1000 \pm 7.5$ meters if the 25 planks that are used are each made from a different mold (*i.e.*, a different mold is used for each plank)?

(d) **2 Points.** Explain in words the difference between the answers in parts (b) and (c). (*Note: Do not just write, for example, 'One has higher probability than the other because the math says so.' Explain why that is, why you expect the results to be as they are.*)

**8. 12 Points.** An elevator is designed to tolerate a maximum weight of 5000 pounds. We are interested in the probability that the elevator is overloaded, meaning that the total weight of all people onboard exceeds the threshold of 5000 pounds. Assume that the weight $W_i$ of person $i$ can be modeled as an exponential variable with parameter $\lambda = 1/150$, so that $f_{W_i} = \lambda e^{-\lambda w}$ for $w \geq 0$, and that the weights of different people are independent. For parts (a) and (b), assume that exactly 26 people climb onto the elevator.

(a) **3 Points.** Using the Markov inequality, compute an upper bound on the probability that the elevator is overloaded.

(b) **4 Points.** Using the Central Limit Theorem, compute an approximation to the probability that the elevator is overloaded. Give a numerical value.

For part (c), suppose that a *random number* $T$ of people board the elevator, where $T$ is a geometric random variable with parameter $q \in (0, 1)$.

(c) **5 Points.** Using the Chebyshev inequality, compute an upper bound on the probability that the elevator is overloaded, as a function of $q$.

**9. 8 Points.** Consider a polling situation where we want to estimate the fraction $\theta$ of the population that goes to a premium coffee place such as Starbucks when they get coffee to go. The polling process consists of collecting $n$ independent sample responses $X_1, ..., X_n$, where $X_i$ is a Bernouilli random variable such that: $X_i = 1$ if the $i^{\text{th}}$ person polled goes to a premium coffee place. Suppose that out of the $n = 1500$ people polled here, 510 of them said that they went to a premium coffee place. We want to estimate $\theta$ with the sample mean $\hat{\Theta}_n$, and construct a 95% confidence interval based on a normal approximation.

(a) **3 Points.** Calculate the unbiased variance estimate $\hat{S}_n^2$.

(b) **3 Points.** Calculate the lower and upper bounds (i.e., the interval $[\hat{\Theta}_n^-, \hat{\Theta}_n^+]$) of the 95% confidence interval using the unbiased variance estimate calculated in part a.

(c) **2 Points.** What does this confidence interval mean/represent? Explain in words.

**10. 6 Points.** Answer the following questions with words, describing qualitatively what is asked.

(a) **2 Points.** Explain in words the differences between Bayesian and Classical inference.

(b) **2 Points.** What is the MAP (Maximum a Posteriori Probability) rule? When/how do you use it?

(c) **2 Points.** Give one important property of the estimator $\hat{\Theta}$ in LMS (Least Mean Square) Estimation.

**11. 6 Points.** The time between successive train arrivals (in minutes) at Grand Central Station in NY City to take you downtown is exponentially distributed with parameter $T$. Your prior PDF of $T$ is given by

$$f_T(t) = \begin{cases} 15t, & \text{for } t \in [0, 1/5], \\ 0, & \text{otherwise.} \end{cases}$$

(a) **4 Points.** You arrive at the train station on Monday morning and you have to wait 20 minutes for the train to arrive. Calculate the posterior PDF (leave any integrals in that form, no need to evaluate the integrals). (*Hint: define a random variable $X$ to be the random wait time and proceed from there.*)

(b) **2 Points.** What is the MAP (Maximum a Posteriori) estimate of $T$?

## USEFUL STUFF

- Unbiased variance estimate: $\hat{S}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\Theta}_n)^2$.

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### Standard Normal Probabilities

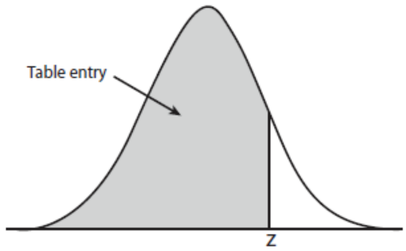


**Table entry for Z is the area under the standard normal curve to the left of Z.**

| | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 2/3 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| **3.1** | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| **3.2** | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| **3.3** | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| **3.4** | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

FIGURE 1. Standard Normal Table.