ECE 3100 **PRELIM 1** **Fall 2014**

Wednesday, 01 October 2014

NAME: ____ **netID:** _____

- Exam duration: 2:55 PM - 4:10 PM, that's 1 hour 15 minutes
- There are 6 questions to this exam
- Exams are collected at 4:10 PM, no extensions
- Write your answers on this set of sheets
- Write your calculations/reasonings *clearly*: no points awarded for what you *intended to write* or *meant*
- Circle your final answer
- Keep your cheat sheet after the exam
- Useful formulae:

$$\frac{\mathrm{d}}{\mathrm{d}r}\sum_{n=1}^{\infty} r^n = \sum_{n=1}^{\infty} n r^{n-1}$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \text{ if } |r| < 1$$

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1. 20 Points (4/question). True or False: Answer each of the following question *clearly* with True or False. *No partial credit.*

(a) If events A^c and B^c are independent from one another, then so are A and B independent from one another.
True: We proved something like this in homework:

$$\begin{aligned} P(A \cup B) &= 1 - P((A \cup B)^c) \\ &= 1 - P(A^c \cup B^c) \\ &= 1 - (P(A^c) + P(B^c) - P(A^c \cap B^c)) \\ &= 1 - (P(A^c) + P(B^c) - P(A^c)P(B^c)), \text{ bc } A^c \text{ and } B^c \text{ are indep.} \\ &= (1 - P(A^c))(1 - P(B^c)) \\ &= P(A)P(B). \end{aligned}$$

(b) If events A and B are disjoint from one another, then so are A^c and B^c disjoint from one another.
False: Consider, for example, $\Omega = \{3, 5, 7, 9\}$, and $A = \{3\}$ and $B = \{9\}$. Then A and B are disjoint but A^c and B^c are not disjoint.

(c) The universal set Ω is independent of itself.
True: $P(\Omega \cap \Omega) = P(\Omega) = 1 = 1 \cdot 1 = P(\Omega) \cdot P(\Omega)$.

(d) If A and B are events, and $P(A) > 0$, then

$$P\big((A \cap B) \ |(A \cup B)\big) \leq P\big((A \cap B) \ |A\big).$$

True: From the definition of conditional probability:

$$\begin{aligned} P(A \cap B | A \cup B) &= \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} \\ &= \frac{P(A \cap B)}{P(A \cup B)} \\ &= \frac{P(A \cap B)}{P(A) + P(B \cap A^c)} \\ &\leq \frac{P(A \cap B)}{P(A)} \\ &= P((A \cap B)|A). \end{aligned}$$

(e) Given the three mutually independent discrete random variables X, Y and Z, then X and $Y + Z$ are not necessarily independent.
False. X **is necessarily independent of** $g(Y, Z)$ **for any function** $g(Y, Z)$**, including** $g(Y, Z) = Y + Z$**.**

2. 10 Points. Let X be a random variable with $E[X]$=1 and var(X)=4. Calculate the following:

(a) $E[2X-4]$
$= \mathbf{2}E[X]-4 =$ **-2.**

(b) $E[X^2]$
$= E[X]^2+$ **var**(X)**=5**

(c) $E[(2X-4)^2]$
$=E[(4X^2+16-16X)] = 4E[X^2]-16E[X]+16 =$**20**

3. 10 Points. A data communication systems sends data packets of fixed length. There is noise in the communication channel: this noise can result in a data packet that is received with errors in it. If this happens, then the data packet is sent again. The probability that a data packet is received with errors in it is q. Calculate the average number of transmissions that are necessary before a data packet is received correctly.

Let N = number of packets that are transmitted until the first success. Then,

$$P(N = n) = q^{n-1}(1-q), \qquad n = 1, 2, 3, \ldots$$

And so,

$$\begin{aligned} E[N] &= \sum_{n=1}^{\infty} n q^{n-1}(1-q) = (1-q)\sum_{n=1}^{\infty} n q^{n-1} \\ &= (1-q)\frac{d}{dq}\sum_{n=1}^{\infty} q^n = (1-q)\frac{d}{dq}\frac{q}{1-q} = \frac{1}{1-q}. \end{aligned}$$

4. 20 Points. Mike and Sophia each need to buy a bicycle. Four blue (for \$300 each), three black (for \$200 each) and two green (for \$100 each) bicycles are available at the bike shop. Sophia randomly picks one of the bikes and buys it. Right after that, Mike does the same thing. Let A be the event that Sophia bought a blue bicycle, and let B be the event that Mike bought a blue bicycle.

(a) What is $P(A)$? What is $P(A \mid B)$?
We have $P(A) = 4/9$ (4 blue bikes out of 9), and $P(A|B) = 3/8$ (since we know that Mike has a blue bike, Sophia can have one of 3 blue bikes out of the remaining 8).

(b) Are A and B independent events? Justify your answer (don't just answer Yes or No).
Since $P(A) \neq P(A|B)$, the events are not independent. Informally, since there is a fixed quantity of blue bikes, if Sophia buys one, then the chances that Mike also buys one are slightly decreased.

(c) What is the probability that at least one of them bought a blue bicycle?
We want $P(A \cup B)$:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A|B) \cdot P(B) \\ &= \frac{4}{4} + \frac{4}{9} - \frac{3}{8} \cdot \frac{4}{9} = \frac{13}{18} = 0.722. \end{aligned}$$

(d) Given that Mike bought a blue bike, what is the expected value of the amount of money that Sophia spent?
If Mike bought a blue bicycle, then the conditional probabilities of Danielle buying a blue, black or green bicycle are 3/8, 3/8 and 2/8, respectively. The expected amount that Danielle will spend is then:

$$\$300 \cdot \frac{3}{8} + \$200 \cdot \frac{3}{8} + \$100 \cdot \frac{2}{8} = \$212.50.$$

5. 20 Points. A random variable has a probability density function (PDF) given by

$$f_X(x) = \begin{cases} ce^{-2x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant. Calculate:

(a) the constant c.
We have that

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \ ,$$

and so,

$$c\int_{0}^{\infty} e^{-2x}dx = 1 \ ,$$

which gives c=2.

(b) $P(X > 2)$.
We have that

$$P(X > 2) = \int_{2}^{\infty} f_X(x)dx \ ,$$

and so

$$P(X > 2) = \int_{2}^{\infty} 2e^{-2x}dx = e^{-4} \ .$$

(c) $P(X < 3)$.
We have that

$$P(X < 3) = \int_{0}^{3} 2e^{-2x}dx = 1 - e^{-6} \ .$$

(d) $P\big((X < 3) \mid (X > 2)\big)$.
We have that

$$P(X < 3 \mid X > 2) = \frac{P(2 < X < 3)}{P(X > 2)} \ ,$$

with the numerator equal to

$$P(2 < X < 3) = \int_{2}^{3} 2e^{-2x}dx = e^{-4} - e^{-6} \ ,$$

and we calculated the denominator in part (b):

$$P(X > 2) = e^{-4} \ ,$$

and so

$$P(X < 3 \mid X > 2) = \frac{e^{-4} - e^{-6}}{e^{-4}} = 1 - e^{-2} \ .$$

6. 20 Points. When Carl goes to Collegetown Bagels (CTB), he buys a Caramel Vanilla Latte with probability p and a plain drip coffee with probability $1-p$. If he buys the Caramel Vanilla Latte, he also gets a pecan diamond dessert with probability q_1. And if instead of the Latte he buys the plain coffee, he buys the same dessert with probability q_2.

(a) Carl went to lunch to CTB yesterday. What was the probability that he bought a pecan dessert?
Let A denote the event that Carl bought a Caramel Vanilla Latte (then A^c denotes the event that Carl does not buy the latte), and let D denote the event that Carl gets a pecan dessert. Then by by the total probability theorem,

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|A^c)P(A^c) \\ &= q_1 p + q_2(1-p). \end{aligned}$$

(b) Given that Carl got a pecan dessert, what is the probability that he bought a Caramel Vanilla Latte?
Using Bayes' rule, $P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{q_1 p}{q_1 p + q_2(1-p)}$.

(c) Given that Carl did not get a pecan dessert, what is the probability that he did not buy a Latte?
Again using Bayes' rule:

$$\begin{aligned} P(A^c|D^c) &= \frac{P(D^c|A^c)P(A^c)}{P(D^c)} \\ &= \frac{(1-P(D|A^c)) \cdot P(A^c)}{1-P(D))} \\ &= \frac{(1-q_2)(1-p)}{1-(q_1 p + q_2(1-p))}. \end{aligned}$$

(d) Under what conditions is the event that Carl gets a pecan dessert independent of him buying a Latte? Interpret your answer.
For A and D to be independent, we must have that $P(A) \cdot P(D) = P(A \cap D)$, *i.e.*,

$$p(q_1 p + q2(1-p)) = q_1 p = q_1 p^2 + q_2 p(1-p),$$

and bringing all terms to the left of the equality:

$$q_1 p - q_1^2 - q_2 p(1-p) = 0,$$

and grouping terms, we get

$$p(1-p)(q_1 - q_2) = 0.$$

So from this result, we need that $p = 0$, or $p = 1$ or $q_1 = q_2$ to have independence. The condition that $q1 = q2$ is sufficient for independence because then whether or not Carl buys a Latte clearly does not affect the probability that he gets a pecan dessert. If $p = 0$ or $p = 1$, then the event A has zero or unit probability, and in either case it is independent of everything - those two cases would be trivial.