(a) 4 point running average

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \quad \text{for} \quad L=4$$

$$Y(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} X(z) = X(z)\frac{1}{L}\underbrace{\sum_{k=0}^{L-1} z^{-k}}_{(*)} = X(z)\frac{1}{L}\sum_{k=0}^{L-1}(z^{-1})^k$$

$$= X(z)\frac{1}{L}\frac{1-(z^{-1})^L}{1-z^{-1}} = X(z)\frac{1}{L}\frac{1-z^{-L}}{1-z^{-1}}$$

$$\Rightarrow H_L(z) = \frac{Y(z)}{X(z)} = \frac{1}{L}\frac{1-z^{-L}}{1-z^{-1}}$$

poles: $z=1$

zeros: $1-z^{-L}=0 \Rightarrow z^L-1=0 \Rightarrow z^L=1 \Rightarrow z^L = e^{j2\pi n} \quad n\in\mathbb{Z}$

$\Rightarrow z = e^{j2\pi n/L} \quad n\in\{0,1,\dots,L-1\}$

The pole @ $z=1$ cancels with the zero @ $z=1$ so really

poles: none

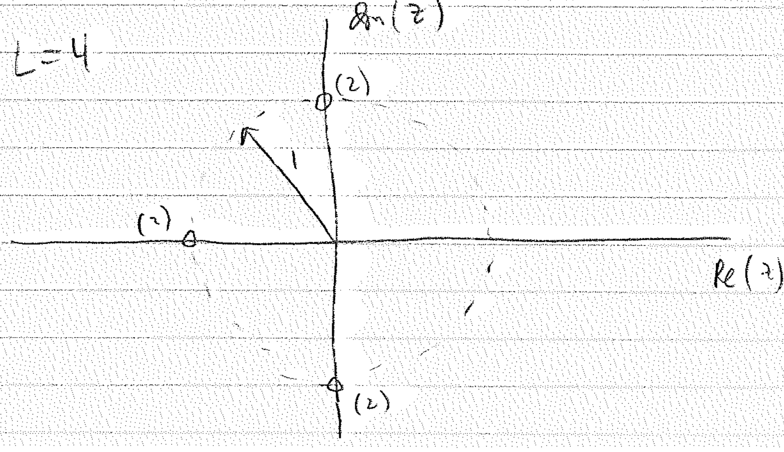
zeros: $z = e^{j2\pi n/L} \quad n\in\{1,2,\dots,L-1\}$ no 0!

$$H(z) = H_L(z)H_L(z) = \frac{1}{L}\frac{1-z^{-L}}{1-z^{-1}}\,\frac{1}{L}\frac{1-z^{-L}}{1-z^{-1}}$$

(b) poles: none

zeros: double zero @ $z = e^{j2\pi n/L} \quad n\in\{1,2,\dots,L-1\}$

L=4



(c) $H_L(\Omega) = H_L(z = e^{j\Omega})$

$$= \frac{1}{L}\,\frac{1-(e^{j\Omega})^{-L}}{1-(e^{j\Omega})^{-1}} = \frac{1}{L}\,\frac{1-e^{-j\Omega L}}{1-e^{-j\Omega}}$$

$$= \frac{1}{L}\,\frac{e^{-j\Omega L/2}\left(e^{+j\Omega L/2} - e^{-j\Omega L/2}\right)}{e^{-j\Omega/2}\left(e^{+j\Omega/2} - e^{-j\Omega/2}\right)}$$

$$= \frac{1}{L}\, e^{-j\Omega(L/2 - 1/2)} \quad \frac{2j \sin \Omega L/2}{2j \sin \Omega/2}$$

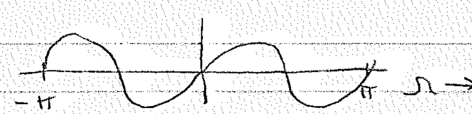
$$= \frac{1}{L}\, e^{-j\Omega(L-1)/2} \quad \frac{\sin \Omega L/2}{\sin \Omega/2}$$

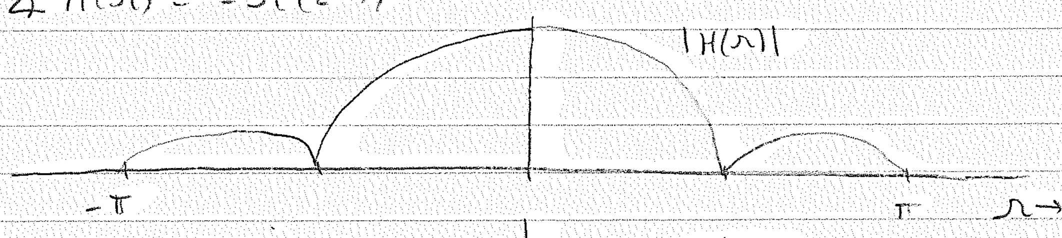
$$H(\Omega) = H_L(\Omega) H_L(\Omega)$$

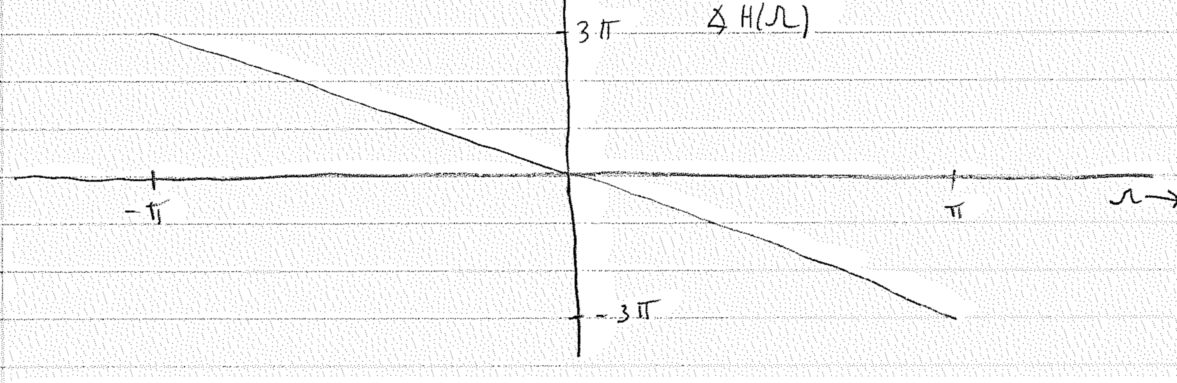
$$= \frac{1}{L^2}\, e^{-j\Omega(L-1)} \left(\frac{\sin \Omega L/2}{\sin \Omega/2}\right)^2$$

(d) $$\left.\begin{aligned} |H(\Omega)| &= \frac{1}{L^2}\left(\frac{\sin \Omega L/2}{\sin \Omega/2}\right)^2 \\ \measuredangle H(\Omega) &= -\Omega(L-1) \end{aligned}\right\} L=2$$

$\sin \Omega L/2$ for $L=4$



e) return to $(*)$ for $L=4$:

$$\underline{Y}(z) = \underline{X}(z)\, \frac{1}{L} \sum_{k=0}^{L-1} z^{-k} = \underline{X}(z)\, \frac{1}{4} \sum_{k=0}^{3} z^{-k}$$

$$\Longrightarrow H_{L=4}(z) = \frac{\underline{Y}(z)}{\underline{X}(z)} = \frac{1}{4} \sum_{k=0}^{3} z^{-k}$$

$$\Longrightarrow H(z) = \left( \frac{1}{4} \sum_{k=0}^{3} z^{-k} \right)$$

$$= \frac{1}{16} \left( 1 + z^{-1} + z^{-2} + z^{-3} \right)\left( 1 + z^{-1} + z^{-2} + z^{-3} \right)$$

$$= \frac{1}{16} \Big[ 1$$
$$+ z^{-1} ( 1 + 1 )$$
$$+ z^{-2} ( 1 + 1 + 1 )$$
$$+ z^{-3} ( 1 + 1 + 1 + 1 )$$
$$+ z^{-4} ( 1 + 1 + 1 )$$
$$+ z^{-5} ( 1 + 1 )$$
$$+ z^{-6} \Big]$$

$$= \frac{1}{16} \left[ 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6} \right]$$

$$\updownarrow$$

$$h[n] = \frac{1}{16} \left[ \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 3\delta[n-4] + 2\delta[n-5] + \delta[n-6] \right]$$

(a) $H(z) = \underbrace{(1 - z^{-1})(1 + z^{-2})(1 + z^{-1})}$

$$= (1 - z^{-2})(1 + z^{-2})$$

$$= 1 - z^{-4}$$

$$\updownarrow$$

$$h[n] = \delta[n] - \delta[n-4]$$

$$\Longrightarrow y[n] = x[n] - x[n-4]$$

(b) $H(\Omega) = H(z = e^{j\Omega}) = 1 - (e^{j\Omega})^{-4} = 1 - e^{-j\Omega 4}$

$$= e^{-j\Omega 2}\left(e^{+j\Omega 2} - e^{-j\Omega 2}\right)$$

$$= e^{-j\Omega 2}\, 2j \sin(\Omega 2)$$

(c) $|H(\Omega)| = 2\,|\sin(\Omega 2)|$

$$\measuredangle H(\Omega) = \begin{cases} -\Omega 2 + \frac{\pi}{2} & \sin(\Omega 2) > 0 \\ -\Omega 2 + \frac{\pi}{2} + \pi & \sin(\Omega 2) < 0 \end{cases}$$

$$= \begin{cases} -\Omega 2 + \frac{\pi}{2} & \sin(\Omega 2) > 0 \\ -\Omega 2 - \frac{\pi}{2} & \sin(\Omega 2) < 0 \end{cases}$$

since $\frac{\pi}{2} + \pi = \frac{3\pi}{2} = \frac{3\pi}{2}$ is equivalent

to $\frac{\pi}{2} + \pi - 2\pi = \frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

(d) zeros of $H(z)$ are

$$1 - z^{-4} = 0 \Longrightarrow z^4 - 1 = 0 \Longrightarrow z^4 = 1 \Longrightarrow z^4 = e^{j2\pi n}$$

$$\Longrightarrow z = e^{j2\pi n/4} = e^{j\pi n/2} \quad n \in \{0, 1, 2, 3\}$$

So the system will null signals at

$$\Omega = n\pi/2 \quad n \in \{0, 1, 2, 3\}.$$

(e) $\begin{cases} X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} \\ x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{+j\Omega n} \, d\Omega \end{cases}$

So, $X_1(\Omega) = 2\pi \delta(\Omega - \Omega_0)$ implies

$$x_1[n] = \frac{1}{2\pi} \int_{2\pi} 2\pi \delta(\Omega - \Omega_0) e^{+j\Omega n} \, d\Omega = e^{j\Omega_0 n}$$

So $x_2[n] = \cos \Omega_0 n = \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)$ has transform

$X_2(\Omega) = \pi [\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$.

$x[n] \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n]$ implies $Y(\Omega) = H(\Omega) X(\Omega)$.

So $Y_1(\Omega) = H(\Omega) X_1(\Omega) = H(\Omega) 2\pi \delta(\Omega - \Omega_0) = H(\Omega_0) 2\pi \delta(\Omega - \Omega_0)$.

Therefore $y_1[n] = H(\Omega_0) e^{j\Omega_0 n} = |H(\Omega_0)| \exp(j(\Omega_0 n + \measuredangle H(\Omega_0)))$.

Similarly, $Y_2(\Omega) = H(\Omega) X_2(\Omega) = H(\Omega) \pi [\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$

$= \frac{1}{2} H(\Omega_0) 2\pi \delta(\Omega - \Omega_0) + \frac{1}{2} H(-\Omega_0) 2\pi \delta(\Omega + \Omega_0)$.

Therefore, $y_2[n] = \frac{1}{2} H(\Omega_0) e^{j\Omega_0 n} + \frac{1}{2} H(-\Omega_0) e^{-j\Omega_0 n}$

$= \frac{1}{2} |H(\Omega_0)| e^{j(\Omega_0 n + \measuredangle H(\Omega_0))} + \frac{1}{2} |H(-\Omega_0)| e^{-j(\Omega_0 n - \measuredangle H(-\Omega_0))}$. $(+)$

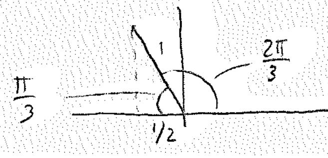
If $h[n]$ is real then

$$H(-\Omega) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j(-\Omega) n} = \left(\sum_{n=-\infty}^{+\infty} h[n] e^{-j\Omega n}\right)^* = H^*(\Omega)$$

$\Longrightarrow |H(\Omega)| = |H(-\Omega)|$ and $\measuredangle H(\Omega) = -\measuredangle H(-\Omega)$.

Therefore $(+)$ implies $y_2[n] = \frac{1}{2} |H(\Omega_0)| e^{j(\Omega_0 n + \measuredangle H(\Omega_0))} + \frac{1}{2} |H(\Omega_0)| e^{-j(\Omega_0 n + \measuredangle H(\Omega_0))}$

$= |H(\Omega_0)| \cos(\Omega_0 n + \measuredangle H(\Omega_0))$

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$\Omega_0 = \pi/3$

$$\Rightarrow |H(\Omega_0)| = 2|\sin(\Omega_0 2)| = 2\left|\sin\left(\overbrace{\frac{\pi}{3}2}^{120^\circ}\right)\right| = 2\,\frac{\sqrt{1-\left(\frac{1}{2}\right)^2}}{1}$$

$$= 2\sqrt{\frac{3}{4}} = 2\,\frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\angle H(\Omega_0) = -\Omega_0 2 + \frac{\pi}{2} = -\frac{\pi}{3}2 + \frac{\pi}{2} \quad \text{since } \sin(2\Omega_0) > 0$$

$$= \left(-\frac{2}{3} + \frac{1}{2}\right)\pi$$

$$= \left(\frac{-4}{6} + \frac{3}{6}\right)\pi$$

$$= -\pi/6$$

Output is

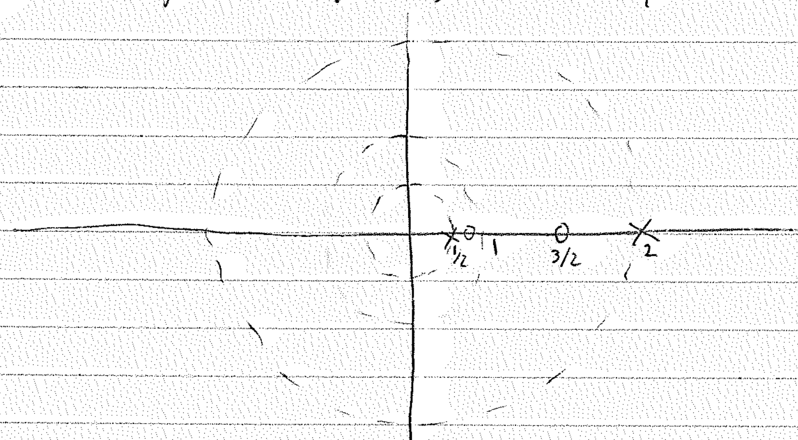
$$\sqrt{3}\cos\left(\frac{\pi}{3}n + \frac{-\pi}{6}\right)$$

$$\left(z - \tfrac{1}{2}\right)(z - 2) = \left(z - \tfrac{3}{4}\right)\left(z - \tfrac{3}{2}\right)$$

$$z^2 - 2.5z + 1 = z^2 - \tfrac{9}{4}z + \tfrac{9}{8}$$

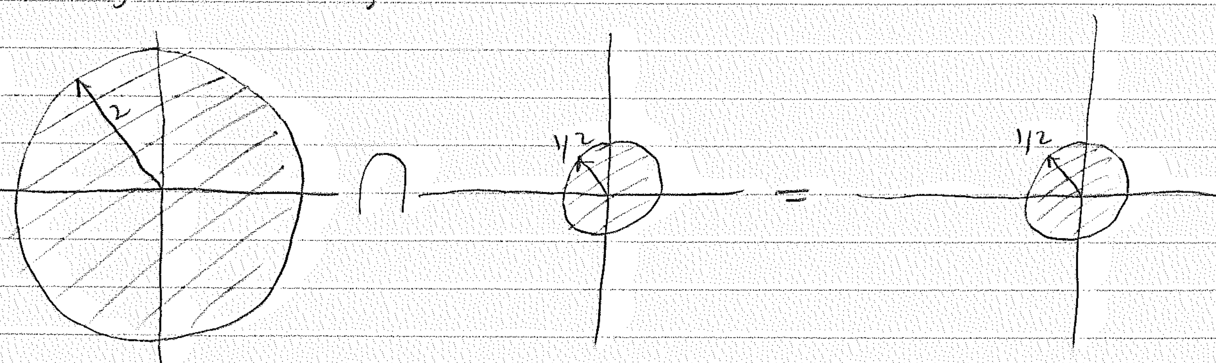
$$1 - 2.5z^{-1} + z^{-2} = 1 - \tfrac{9}{4}z^{-1} + \tfrac{9}{8}z^{-2}$$

$$y[n] - 2.5y[n-1] + y[n-2] = x[n] - \tfrac{9}{4}x[n-1] + \tfrac{9}{8}x[n-2] \quad (*)$$

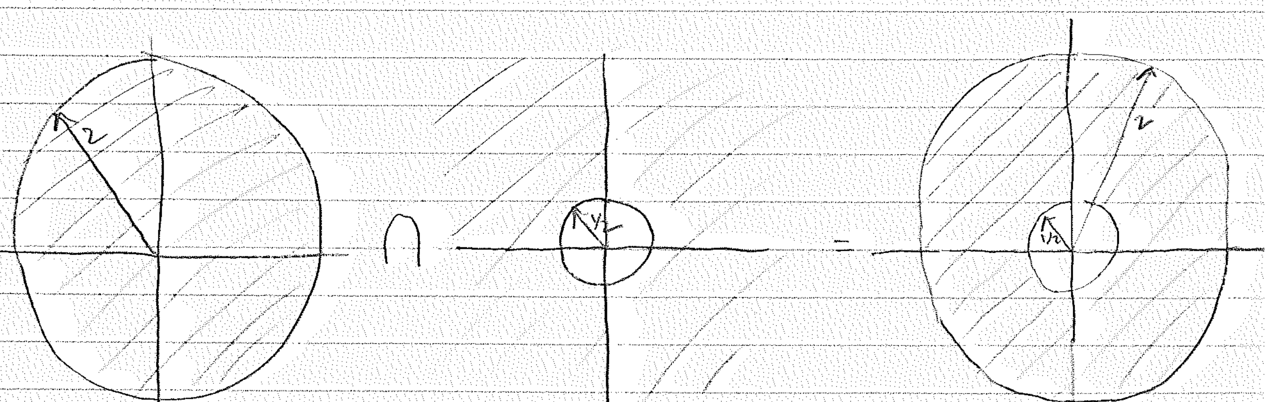



Possible regions of convergence

(1)

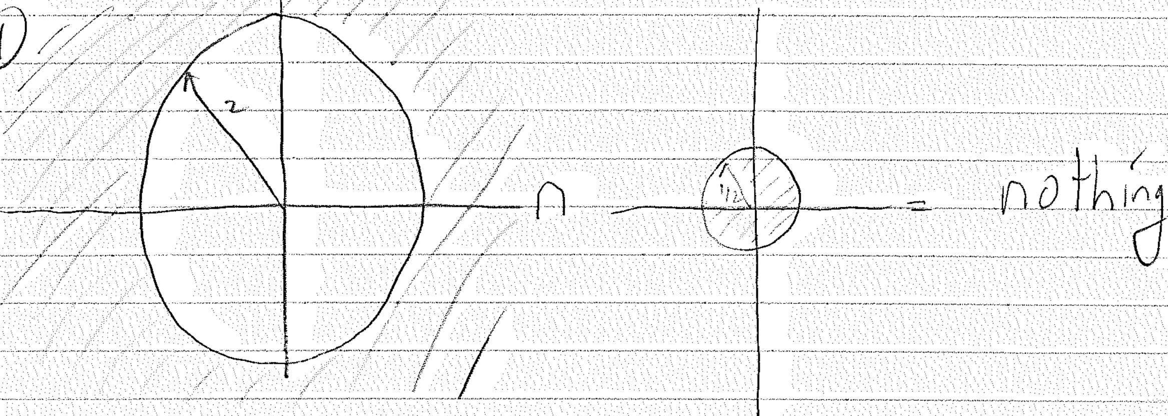



(2)



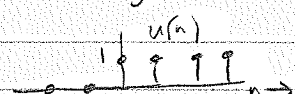
(3)


(4)




So there are three possible regions of convergence.

$x(n) = a^n u(n)$

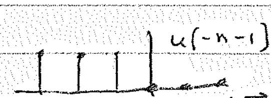



$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} \quad |az^{-1}| < 1$$

$$\Updownarrow$$

"outside a circle" $\rightsquigarrow$ $|a| < |z|$

$x(n) = -a^n u(-n-1)$




$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^{m} = -\sum_{m=1}^{\infty} (a^{-1}z)^m$$

$m = -n$

$$= 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m = 1 - \frac{1}{1-a^{-1}z} \quad |a^{-1}z| < 1$$

$$\Updownarrow$$

"inside a circle" $\rightsquigarrow$ $|z| < |a|$

$$= \frac{1-a^{-1}z-1}{1-a^{-1}z} \ (|z|<|a|) = \frac{-a^{-1}z}{1-a^{-1}z} \ (|z|<|a|) = \frac{-1}{az^{-1}-1} \ (|z|<|a|)$$

$$= \frac{1}{1-az^{-1}} \quad |z| < |a|$$

Summary

$$x(n) = a^n u(n) \longleftrightarrow X(z) = \frac{1}{1-az^{-1}} \quad |a| < |z|$$

$$x(n) = -a^n u(-n-1) \longleftrightarrow X(z) = \frac{1}{1-az^{-1}} \quad |z| < |a|$$

The two answers differ only in the region of convergence.

Take z transform of Eq (*):

$$Y(z) - 2.5z^{-1} Y(z) + z^{-2} Y(z) = X(z) - \frac{9}{4} z^{-1} X(z) + \frac{9}{8} z^{-2} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{9}{4} z^{-1} + \frac{9}{8} z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

degree of numerator $\geq$ degree of denominator. Therefore, perform synthetic division:

$$\begin{array}{r|l} & \frac{9}{8} \\ \hline 1 - 2.5z^{-1} + z^{-2} & 1 - \frac{9}{4} z^{-1} + \frac{9}{8} z^{-2} \\ & \frac{9}{8} - \frac{5}{2}\frac{9}{8} z^{-1} + \frac{9}{8} z^{-2} \\ \hline & -\frac{1}{8} + \frac{9}{16} z^{-1} + 0 z^{-2} \end{array}$$

$$-\frac{9}{4} + \frac{45}{16} = \frac{-36+45}{16} = \frac{9}{16}$$

So
$$H(z) = \frac{9}{8} + \frac{-\frac{1}{8} + \frac{9}{16} z^{-1}}{1 - 2.5z^{-1} + z^{-2}}$$

$$= \frac{9}{8} + \frac{-\frac{1}{8} + \frac{9}{16} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - 2z^{-1})}$$

Partial fractions expansion:

$$\frac{-\frac{1}{8} + \frac{9}{16} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - 2z^{-1})} = \frac{A}{1 - \frac{1}{2} z^{-1}} + \frac{B}{1 - 2z^{-1}}$$

$$A = \left. \frac{-\frac{1}{8} + \frac{9}{16} z^{-1}}{1 - 2z^{-1}} \right|_{z=\frac{1}{2}} = \frac{-\frac{1}{8} + \frac{9}{16} 2}{1 - (2)(2)} = \frac{\frac{-2+18}{16}}{-3} = \frac{-1}{3}$$

$$B = \left.\frac{-\frac{1}{8} + \frac{9}{16}z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=2} = \frac{-\frac{1}{8}+\frac{9}{16}\frac{1}{2}}{1-\frac{1}{2}\frac{1}{2}} = \frac{\frac{-4+9}{32}}{\frac{3}{4}} = \frac{5}{32}\,\frac{4}{3} = \frac{5}{24}$$

So

$$H(z) = \frac{9}{8} + -\frac{1}{3}\,\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{5}{24}\,\frac{1}{1-2z^{-1}}$$

Case ①:

$$h[n] = \frac{9}{8}\delta[n] - \frac{1}{3}\left(-\left(\frac{1}{2}\right)^n u[-n-1]\right) + \frac{5}{24}\left(-2^n u[-n-1]\right)$$

$$= \frac{9}{8}\delta[n] + \frac{1}{3}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{5}{24}\,2^n u[-n-1]$$

$h[n] = 0$ for $n > 0$ — anti causal.

$\left.\begin{array}{l}\left(\frac{1}{2}\right)^n \text{ blows up as } n \to -\infty \\ 2^n \text{ rapidly dies away as } n \to -\infty\end{array}\right\}$ so $\sum_{n=-\infty}^{+\infty} |h[n]| = \infty$ — not BIBO stable

Case ②:

$$h[n] = \frac{9}{8}\delta[n] - \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{5}{24}\left(-2^n u[-n-1]\right)$$

$$= \frac{9}{8}\delta[n] - \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{5}{24}\,2^n u[-n-1]$$

$h[n]$ is not zero for $n>0$ or $n<0$ — non causal

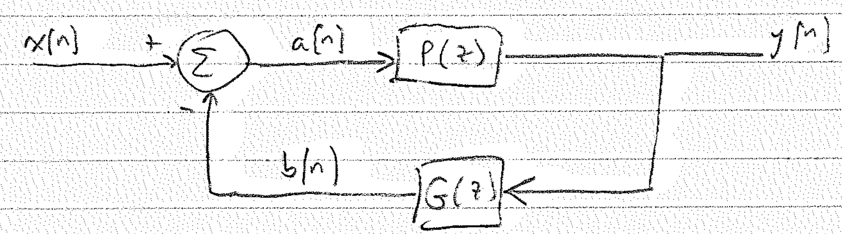
$\left.\begin{array}{l}\left(\frac{1}{2}\right)^n \text{ rapidly dies away as } n \to +\infty \\ 2^n \text{ rapidly dies away as } n \to -\infty\end{array}\right\}$ so $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$ — is BIBO stable

Case ③:

$$h[n] = \frac{9}{8}\delta[n] - \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{5}{24}\,2^n u[n]$$

$h[n]$ is zero for $n<0$ — causal

$\left.\begin{array}{l}\left(\frac{1}{2}\right)^n \text{ rapidly dies away as } n \to +\infty \\ 2^n \text{ blows up as } n \to +\infty\end{array}\right\}$ so $\sum_{n=-\infty}^{+\infty} |h[n]| = \infty$ — not BIBO stable.




$$Y(z) = P(z)A(z) = P(z)\left[X(z) - B(z)\right] = P(z)\left[X(z) - G(z)Y(z)\right]$$

solve for $Y(z)$:

$$Y(z) = P(z)X(z) - P(z)G(z)Y(z)$$

$$Y(z)\left[1 + P(z)G(z)\right] = P(z)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{P(z)}{1 + P(z)G(z)} \quad \text{or} \quad = \frac{1}{\frac{1}{P(z)} + G(z)}$$

ROC could become complicated because if $P(z_x)G(z_x) = -1$ then

there is a new pole at $z_x$.

Consider $p[n] = \alpha^n u[n] \longleftrightarrow P(z) = \frac{1}{1-\alpha z^{-1}}$ $|\alpha| < |z|$ with $|\alpha| > 1$

Therefore $p[n]$ is not BIBO stable.

Consider $g[n] = \beta\delta[n] \longleftrightarrow G(z) = \beta$ for all $z$.

$$H(z) = \frac{\frac{1}{1-\alpha z^{-1}}}{1 + \frac{1}{1-\alpha z^{-1}}\beta} = \frac{1}{1 - \alpha z^{-1} + \beta} = \frac{1}{(1+\beta) - \alpha z^{-1}}$$

$$= \frac{1}{1+\beta} \; \frac{1}{1 - \frac{\alpha}{1+\beta} z^{-1}}$$

For causality, the ROC is $\left|\frac{\alpha}{1+\beta}\right| < |z|$.

For BIBO stability, given this ROC, the condition is $\left|\frac{\alpha}{1+\beta}\right| < 1$

which is equivalent to

$$|\alpha| < |1+\beta|$$

If $\beta > -1$ this is equivalent to

$$|\alpha| - 1 < \beta$$