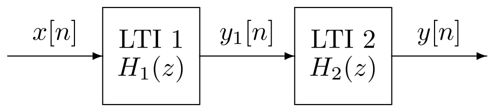
ECE 2200 and ENGRD 2220
Signals and Systems
Spring 2016
Problem Set 9
Due Friday 29, 2016 at 5:00PM.
Location to turn in: "ECE 2200" box on Phillips Hall 2nd floor

1. Slight modification of McClellan, Schafer, Yoder Problem P-7.9. The system described by the block diagram below is constructed out of two LTI systems:



Each system, $h_1[n] \leftrightarrow H_1(z)$ and $h_2[n] \leftrightarrow H_2(z)$ is a 4-point running average, i.e., the $L = 4$ case of

$$y[n] = \frac{1}{L} \sum_{k=0}^{L-1} x[n-k] \tag{113}$$

which is the first equation in Section 7-7.1 on p. 181 with the modification of multiplication by $1/L$.

   (a) Determine the system function $H(z) = H_1(z)H_2(z)$ for the overall system.
   (b) Plot the poles and zeros of $H(z)$ in the z-plane.
   (c) From the system function $H(z)$ obtain an expression for the frequency response $H(\Omega)$ of the overall system.
   (d) Sketch the magnitude and phase of the frequency response $H(\Omega)$ of the overall system for $-\pi \leq \Omega \leq \pi$.
   (e) Use multiplication of z-transform polynomials to determine the impulse response $h[n]$ of the overall system.

2. McClellan, Schafer, Yoder Problem P-7.12. Suppose that a system is defined by the system function

$$H(z) = (1 - z^{-1})(1 + z^{-2})(1 + z^{-1}). \tag{114}$$

   (a) Write the time-domain description of this system in the form of a difference equation.
   (b) Write a formula for the frequency response $H(\Omega)$ of this system.
   (c) Derive simple formulas for the magnitude and phase of the frequency response $H(\Omega)$ as a function of Ω. These formulas must contain no complex terms and no square roots.
   (d) This system can null certain input signals. For which input frequencies Ω_0 is the response to $x[n] = \cos(\Omega_0 n)$ equal to zero, i.e., $x[n]$ is nulled?
   (e) When the input to the system is $x[n] = \cos(\pi n/3)$, determine the output signal $y[n]$ in the form $A\cos(\Omega_0 n + \phi)$, i.e., give numerical values for A, Ω_0, and ϕ.

3. The z transform is a very useful tool for studying difference equations. Often difference and differential equations are used to describe causal systems and only the causal solution is of interest. This is the "initial condition" problem of a differential equations course. But both difference and differential equations describe more than just the causal system. For instance, "backwards" solutions and "two-point boundary value" solutions. One way in which to think about the problem is the ROC of the transform: different choices of ROC give different solutions.

   Consider the difference equation

$$y[n] - 2.5y[n-1] + y[n-2] = x[n] - \frac{9}{4}x[n-1] + \frac{9}{8}x[n-2]. \tag{115}$$

(a) Compute the system function for Eq. 115, i.e., $H(z) = Y(z)/X(z)$.

(b) Compute the poles and zeros of the system function. (Hint: Poles at 1/2 and 2, zeros at 3/4 and 3/2).

(c) Plot the poles and zeros of the system function in the complex z plane.

(d) A ROC is an annular region, because it is a condition on $|z|$, bounded by a pole, because the pole locations are the values of z where the system function is infinite. On three different plots, show the three possible ROCs for Eq. 115.

(e) i. Let $x[n] = a^n u[n]$. Compute the z transform $X(z)$ including the ROC.

ii. Let $x[n] = -a^n u[-n-1]$. Compute the z transform $X(z)$ including the ROC.

Notice that $X(z)$ is the same, but the ROCs are different!

(f) Use synthetic division and partial fractions expansion to express the system function in the form

$$H(z) = c + \frac{A}{1 - az^{-1}} + \frac{B}{1 - bz^{-1}}. \tag{116}$$

(g) What is the ROC for the term c?

(h) What are the two possible ROCs for the term

$$\frac{A}{1 - az^{-1}}? \tag{117}$$

(i) What are the two possible ROCs for the term

$$\frac{B}{1 - bz^{-1}}? \tag{118}$$

(j) The ROC for $H(z)$ has to be the intersection of the ROCs for each term. Counting gives $1 \times 2 \times 2 = 4$ ROCs for $H(z)$. Why did you only find 3 ROCs in Item 3d?

(k) For each of the 3 ROCs in Item 3d:

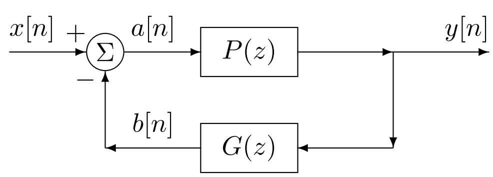
i. Write down the ROCs for each of the terms in Eq. 116.

ii. Determine $h[n]$, the impulse response, which is the inverse z transform of $H(z)$.

iii. Is $h[n]$ BIBO stable?

iv. Is $h[n]$ causal (i.e., $h[n] = 0$ for $n < 0$), anti-causal (i.e., $h[n] = 0$ for $n > 0$), or non-causal (i.e., there are values of n that are both positive and negative for which $h[n]$ is nonzero).

4. Consider the following block diagram where $P(z)$ is the "plant" and $G(z)$ is the "feedback controller":



The Σ with $+$ and $-$ signs means that $a[n] = x[n] - b[n]$.

The problem is that the plant, $P(z)$, is BIBO unstable. The goal is to stablize the transformation from $x[n]$ to $y[n]$ by using the feedback controller $G(z)$.

(a) Express $Y(z)$ in terms of $P(z)$, $X(z)$, and $B(z)$. Express $B(z)$ in terms of $G(z)$ and $Y(z)$. Combine these two equations to express $Y(z)$ in terms of $P(z)$, $G(z)$, $X(z)$, and $Y(z)$. Solve this expression for $Y(z)$ in order to determine the system function $H(z) = Y(z)/X(z)$. Hint: $H(z) = P(z)/[1 + P(z)G(z)]$. The ROC could be complicated because if $P(z_*)G(z_*) = -1$ then there is a new pole at z_*. It is important to keep in mind that you seek causal systems.

(b) Consider $p[n] = \alpha^n u[n]$ with z transform $P(z) = 1/(1 - \alpha z^{-1})$ and ROC $|\alpha| < |z|$ with $|\alpha| > 1$. Consider $g[n] = \beta\delta[n]$ with z transform $G(z) = \beta$ for all z. Is $p[n]$ BIBO stable? The feedback controller $g[n]$ is just an amplifier with gain β.

(c) What is the system function $H(z)$ for this choice of $P(z)$ and $G(z)$? Hint: $H(z)$ has the form $\gamma/(1 - \delta z^{-1})$ for some γ and some δ.

(d) Remembering that this is a causal system, what is the ROC?

(e) Given the ROC, what is the condition on δ so that the overall system (i.e., $x[n]$ to $y[n]$) is BIBO stable?

(f) If $\beta > 0$, what is the condition on β so that the overall system is BIBO stable?