ECE 2200 / Spring 2016 / Problem Set 8 / Solution

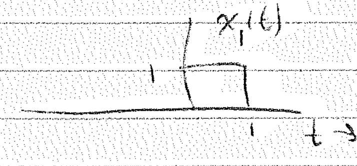
Exercise 12.12 p. 364

Half wave rectifier is defined by

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{Half wave} \\ \text{rectifier} \\ \mathcal{D}\{x(t)\}\end{array}} \longrightarrow y(t)$$

with
$$y(t) = \begin{cases} x(t), & x(t) > 0 \\ 0, & x(t) \leq 0 \end{cases}$$

Suppose
$$x_1(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} =$$



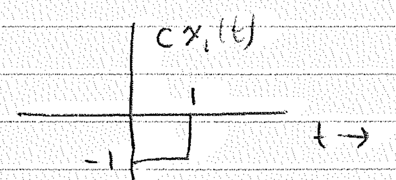
Then $y_1(t) = \mathcal{D}\{x_1(t)\} = x_1(t)$.

Part of the requirements for linearity is that if $x_1(t) \rightarrow y_1(t)$

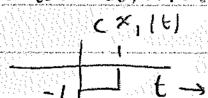
Then $c\,x_1(t) \rightarrow c\,y_1(t)$ for any constant $c \in \mathbb{C}$.

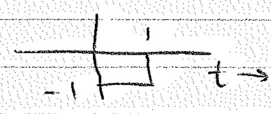
Consider the $x_1(t)$ defined above with the constant $c = -1$.

$$c\,x_1(t) = \begin{cases} -1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} =$$



But $\mathcal{D}\{c\,x_1(t)\}$ = half wave rectifier applied to $c\,x_1(t)$

$= \mathcal{D}\{$  $\} = 0 \neq c\,y_1(t) = (-1)\,x_1(t)$

$=$ 

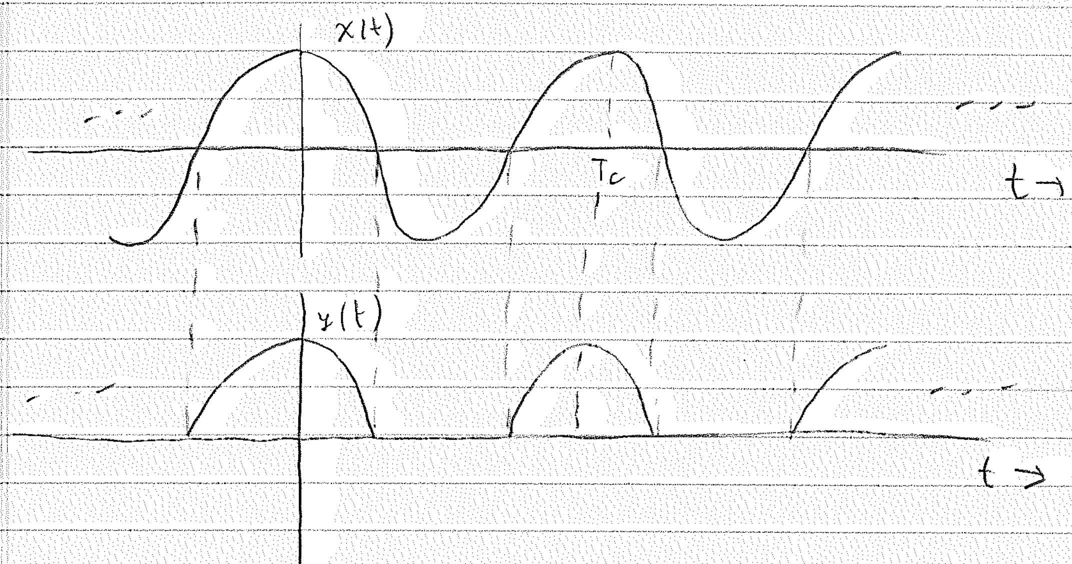
So the system is not linear.

$x(t) = \cos(2\pi f_c t)$

$y(t) = \mathcal{D}\{x(t)\}$ where $\mathcal{D}\{\cdot\}$ is the half wave rectifier defined by

$$\mathcal{D}\{x(t)\} = \begin{cases} x(t) & \text{if } x(t) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$T_c = 1/f_c$



$x(t)$ and $y(t)$ both have period $T_c = 1/f_c$

$$a_k = \frac{1}{T_c}\int_{T_c} y(t)\, e^{-j\frac{2\pi}{T_c}kt}\,dt = \frac{1}{T_c}\int_{-T_c/4}^{+T_c/4} \cos(2\pi f_c t)\, e^{-j\frac{2\pi}{T_c}kt}\,dt$$

$$= \frac{1}{T_c}\int_{-T_c/4}^{+T_c/4} \frac{1}{2}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right) e^{-j2\pi f_c kt}\,dt$$

$$= \frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} e^{-j2\pi f_c(k-1)t}\,dt + \frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} e^{-j2\pi f_c(k+1)t}\,dt$$

$k=+1$ $$\frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} 1\,dt + \frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} e^{-j2\pi f_c 2t}\,dt$$

$$= \frac{f_c}{2}\left(T_c/4 - -T_c/4\right) + \frac{f_c}{2}\,\frac{1}{-j2\pi f_c 2}\left[e^{-j2\pi f_c 2T_c/4} - e^{-j2\pi f_c 2(-T_c/4)}\right]$$

$$= \frac{f_c}{2}\,\frac{T_c}{2} + \frac{f_c}{2}\,\frac{1}{-j2\pi f_c 2}\left[\underbrace{e^{-j\pi}}_{1} - \underbrace{e^{+j\pi}}_{1}\right] = \frac{1}{4}$$

$k=-1$ – very similar to $k=+1$:

$$\frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} e^{+j2\pi f_c 2t}\,dt + \frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} 1\,dt$$

$$= \frac{f_c}{2}\frac{1}{j2\pi f_c 2}\left[e^{j2\pi f_c 2 T_c/4} - e^{j2\pi f_c 2(-T_c/4)}\right] + \frac{f_c}{2}\left(T_c/4 - -T_c/4\right)$$

$$= \frac{f_c}{2}\frac{1}{j2\pi f_c 2}\left[\underbrace{e^{j\pi}}_{1} - \underbrace{e^{-j\pi}}_{1}\right] + \frac{f_c}{2}\frac{T_c}{2} = \frac{1}{4}$$

$k \neq \pm 1$

$$\frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} e^{-j2\pi f_c(k-1)t}\,dt + \frac{f_c}{2}\int_{-T_c/4}^{+T_c/4} e^{-j2\pi f_c(k+1)t}\,dt$$

$$= \frac{f_c}{2}\frac{1}{-j2\pi f_c(k-1)}\left[e^{-j2\pi f_c(k-1)T_c/4} - e^{-j2\pi f_c(k-1)(-T_c/4)}\right]$$

$$+ \frac{f_c}{2}\frac{1}{-j2\pi f_c(k+1)}\left[e^{-j2\pi f_c(k+1)T_c/4} - e^{-j2\pi f_c(k+1)(-T_c/4)}\right]$$

$$= \frac{1}{-j4\pi}\left\{\frac{1}{k-1}\left[e^{-j2\pi(k-1)/4} - e^{+j2\pi(k-1)/4}\right] + \frac{1}{k+1}\left[e^{-j2\pi(k+1)/4} - e^{+j2\pi(k+1)/4}\right]\right\}$$

$$e^{+j2\pi/4} = j, \quad e^{-j2\pi/4} = -j$$

$$= \frac{1}{-j4\pi}\left\{\frac{1}{k-1}\left[je^{-j2\pi k/4} - -je^{+j2\pi k/4}\right] + \frac{1}{k+1}\left[-je^{-j2\pi k/4} - je^{+j2\pi k/4}\right]\right\}$$

$$= \frac{1}{4\pi}\left\{\frac{-1}{k-1}\left[\underbrace{e^{-j2\pi k/4} + e^{+j2\pi k/4}}_{2\cos(2\pi k/4)}\right] + \frac{1}{k+1}\left[\underbrace{e^{-j2\pi k/4} + e^{+j2\pi k/4}}_{2\cos(2\pi k/4)}\right]\right\}$$

$$= \frac{1}{4\pi}2\cos(2\pi k/4)\left\{\underbrace{\frac{-1}{k-1} + \frac{1}{k+1}}\right\} \qquad \frac{-(k+1)+(k-1)}{(k-1)(k+1)} = \frac{-2}{k^2-1}$$

$$= \frac{\cos(2\pi k/4)}{\pi(1-k^2)} \qquad \text{So } a_k = \begin{cases} 1/4 & k = \pm 1 \\ \dfrac{\cos(2\pi k/4)}{\pi(1-k^2)} & k \neq \pm 1 \end{cases}$$

$$H(\omega) = \frac{86 + 28j\omega - 2\omega^2}{24 + 10j\omega - \omega^2}$$

order of numerator polynomial in ω = order of denominator polynomial in ω so first must do synthetic division:

$$\begin{array}{r|l} & \quad 2 \\ \hline 24 + 10j\omega - \omega^2 & 86 + 28j\omega - 2\omega^2 \\ & \underline{48 + 20j\omega - 2\omega^2} \\ & 38 + 8j\omega - 0\omega^2 \end{array}$$

$$\Longrightarrow \quad H(\omega) = 2 + \frac{38 + 8j\omega}{24 + 10j\omega - \omega^2}$$

$$= 2 + \frac{38 + 8j\omega}{(4 + j\omega)(6 + j\omega)}$$

$$= 2 + \frac{A}{4 + j\omega} + \frac{B}{6 + j\omega}$$

Determine the values of A and B by the partial fractions procedure.

$$H(\omega) = \frac{86 + 28j\omega - 2\omega^2}{24 + 10j\omega - \omega^2} = 2 + \frac{38 + 8j\omega}{(4 + j\omega)(6 + j\omega)} = 2 + \frac{A}{4 + j\omega} + \frac{B}{6 + j\omega}$$

Therefore

$$\frac{38 + 8j\omega}{(4 + j\omega)(6 + j\omega)} = \frac{A}{4 + j\omega} + \frac{B}{6 + j\omega}$$

1. determine A: multiply both sides by $4 + j\omega$ and evaluate @ $\omega = -4/j$:

$$\left.\frac{38 + 8j\omega}{6 + j\omega}\right|_{\omega = -4/j} = \left. A + \frac{B(4 + j\omega)}{6 + j\omega}\right|_{\omega = -4/j}$$

$$\Longrightarrow \quad A = \frac{38 + 8j(-4/j)}{6 + j(-4/j)} = \frac{6}{2} = 3$$

(2) determine B: multiply both sides by $6+j\omega$ and evaluate @ $\omega = -6/j$:

$$\left.\frac{38+8j\omega}{4+j\omega}\right|_{\omega=-6/j} = \left.\frac{A(6+j\omega)}{4+j\omega}\right|_{\omega=-6/j} + B$$

$$\Rightarrow B = \frac{38+8j(-6/j)}{4+j(-6/j)} = \frac{-10}{-2} = 5$$

Therefore,

$$H(\omega) = 2 + \frac{3}{4+j\omega} + \frac{5}{6+j\omega}$$

$$h(t) = 2\delta(t) + 3e^{-4t}u(t) + 5e^{-6t}u(t)$$

If $x(t) \longleftrightarrow X(\omega)$ is the input and $y(t) \longleftrightarrow Y(\omega)$

is the output and

$$\frac{d^2 y}{dt^2} + 10\frac{dy}{dt} + 24y(t) = 2x(t) + 8\frac{dx}{dt}$$

then

$$(j\omega)^2 Y(\omega) + 10j\omega Y(\omega) + 24Y(\omega) = 2X(\omega) + 8j\omega X(\omega)$$

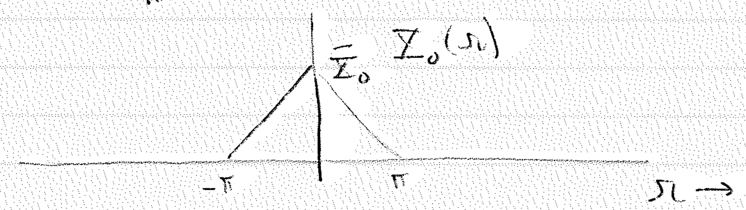
$$\Rightarrow \left[(j\omega)^2 + 10j\omega + 24\right] Y(\omega) = (2 + 8j\omega) X(\omega)$$

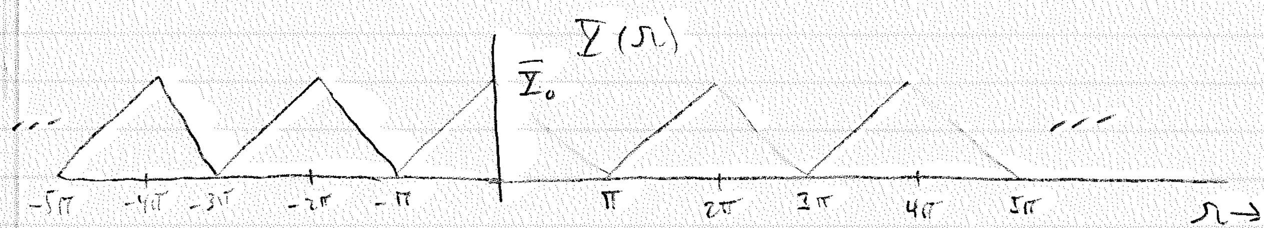
$$\Rightarrow \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2 + 8j\omega}{(j\omega)^2 + 10j\omega + 24}$$

(a) Nothing to do since the answer is given and explained

(b) $$\bar{X}_o(\Omega) = \begin{cases} \bar{X}_o(1 - |\Omega|/\pi) & -\pi \le \Omega \le +\pi \\ 0 & \end{cases}$$

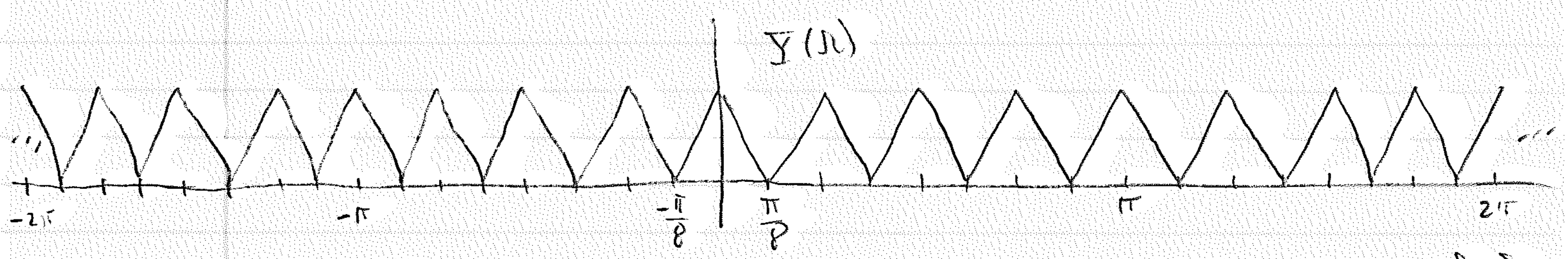
$$X(\Omega) = \sum_{k=-\infty}^{+\infty} X_o(\Omega + k2\pi)$$





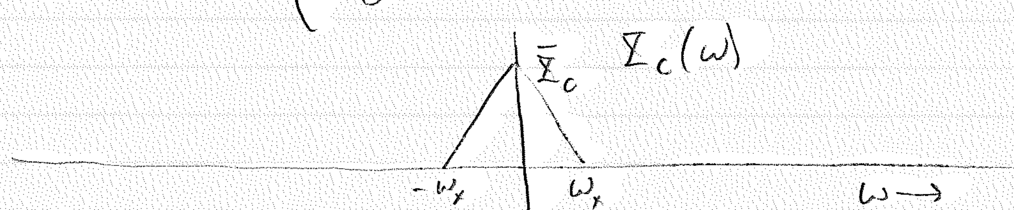
periodic with period 2π

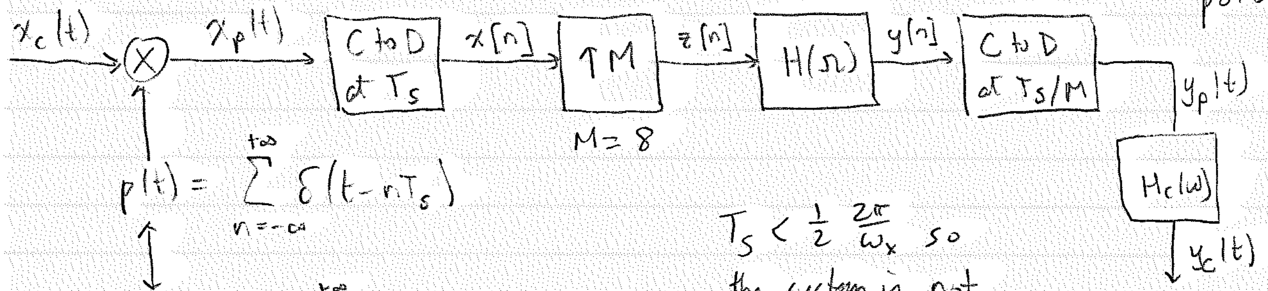
$$Y(\Omega) = X(M\Omega) \qquad M = 8$$



periodic with period $\frac{\pi}{4}$ but it is only required to be periodic with period 2π

(c) $$\bar{X}_c(\omega) = \begin{cases} \bar{X}_c\left(1 - \frac{|\omega|}{\omega_x}\right) & -\omega_x \le \omega \le \omega_x \\ 0 & \end{cases}$$





$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$\updownarrow$$

$$P(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right)$$

$T_s < \frac{1}{2} \frac{2\pi}{\omega_x}$ so the system is not aliased

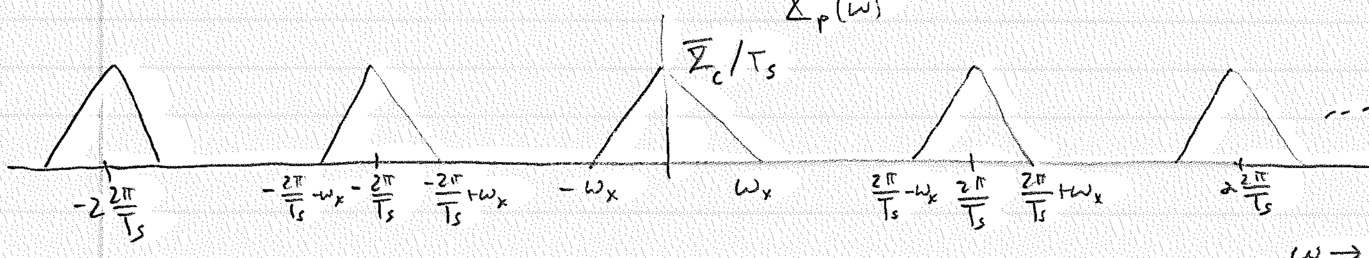
(a) two ways to write $x_p(t) \longleftrightarrow X_p(\omega)$

(i) $x_p(t) = x_c(t)\, p(t)$

$$\updownarrow$$

$$X_p(\omega) = \frac{1}{2\pi} X_c(\omega) * P(\omega) = \frac{1}{2\pi} X_c(\omega) * \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X_c(\omega) * \delta\left(\omega - \frac{2\pi}{T_s} n\right) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X_c\left(\omega - \frac{2\pi}{T_s} n\right)$$



(ii) $x_p(t) = x_c(t)\, p(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} x_c(t)\, \delta(t - nT_s)$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, \delta(t - nT_s)$$

$$\updownarrow$$

$$X_p(\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, \mathcal{F}\{\delta(t - nT_s)\}$$

continuous time Fourier transform delay theorm

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s} \quad 1$$

continuous time Fourier transform of the $\delta(t)$ function.

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s} \qquad (*)$$

difficult to plot using this formula, but it is good for relating the DTFT of $x[n]$ to $x_c(t)$.

(β) $x[n] = x_c(nT_s)$

$\updownarrow$

$$\mathbb{X}(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\Omega n}$$

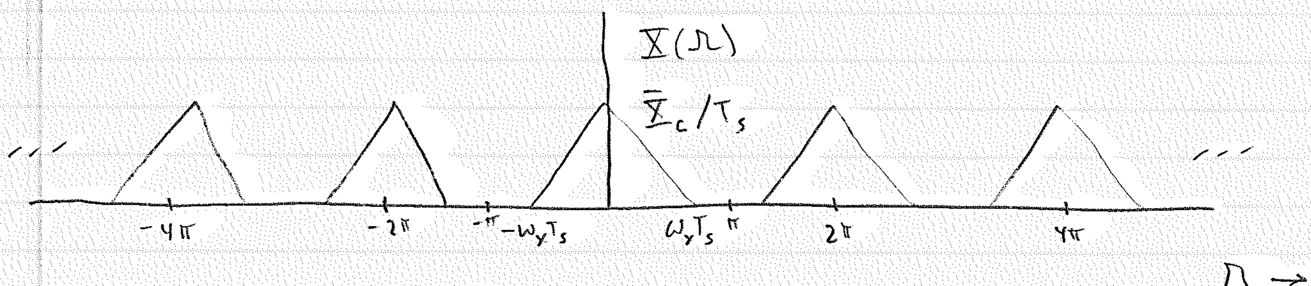
Compare with Eq (*) which is

sam

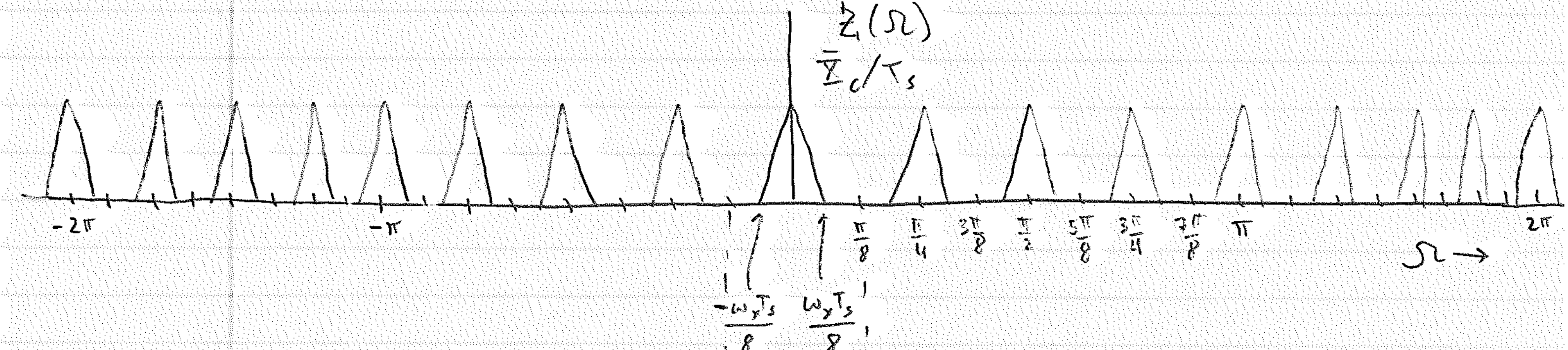
$$\mathbb{X}_p(\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega T_s n}$$

See that

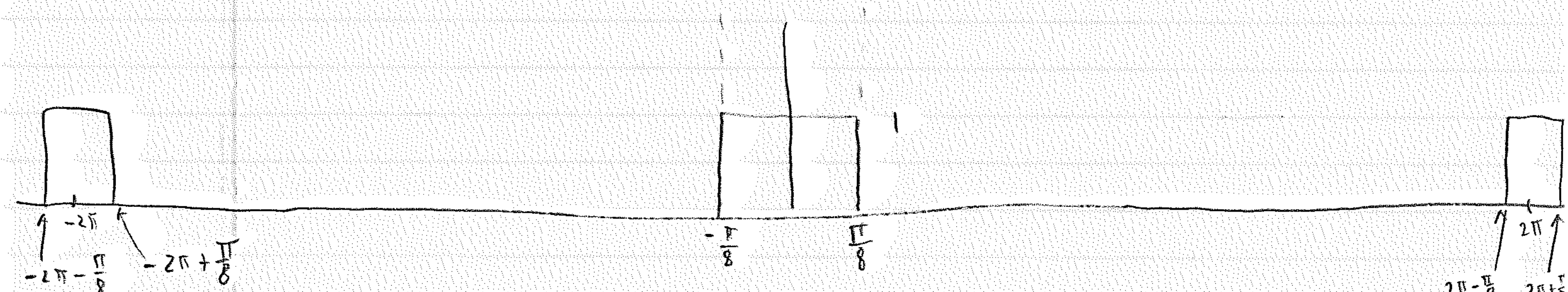
$$\mathbb{X}(\Omega) = \mathbb{X}_p\left(\omega = \frac{\Omega}{T_s}\right)$$



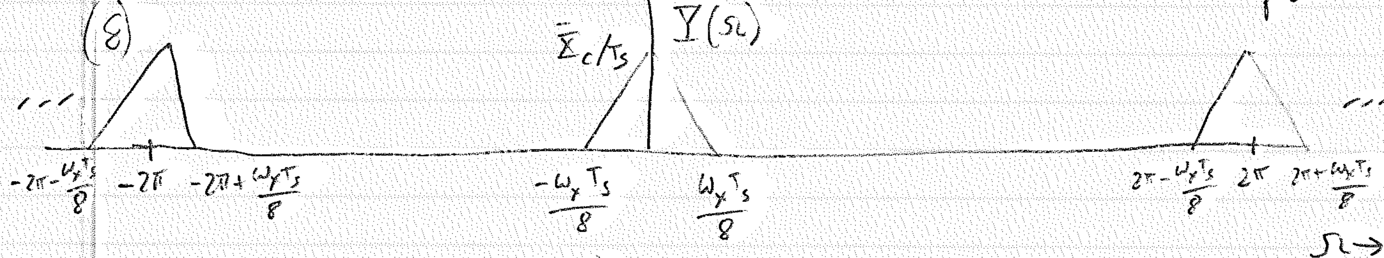
since $T_s < \frac{1}{2}\frac{2\pi}{\omega_x}$ it follows that $\omega_x T_s < \frac{1}{2} 2\pi = \pi$



(γ) $H_0(\Omega) = \begin{cases} 1 & -\frac{\pi}{M} \leq \omega \leq \frac{\pi}{M} \\ 0 & \text{otherwise} \end{cases}$ $\quad H(\Omega) = \sum_{h=-\infty}^{+\infty} H(\Omega + h2\pi)$



(ε)



(ε) $Y(\Omega) = \sum_{n=-\infty}^{+\infty} y[n] e^{-j\Omega n}$

$$y_p(t) = \sum_{n=-\infty}^{+\infty} y[n]\, \delta\left(t - n\frac{T_s}{M}\right)$$

$$\updownarrow$$

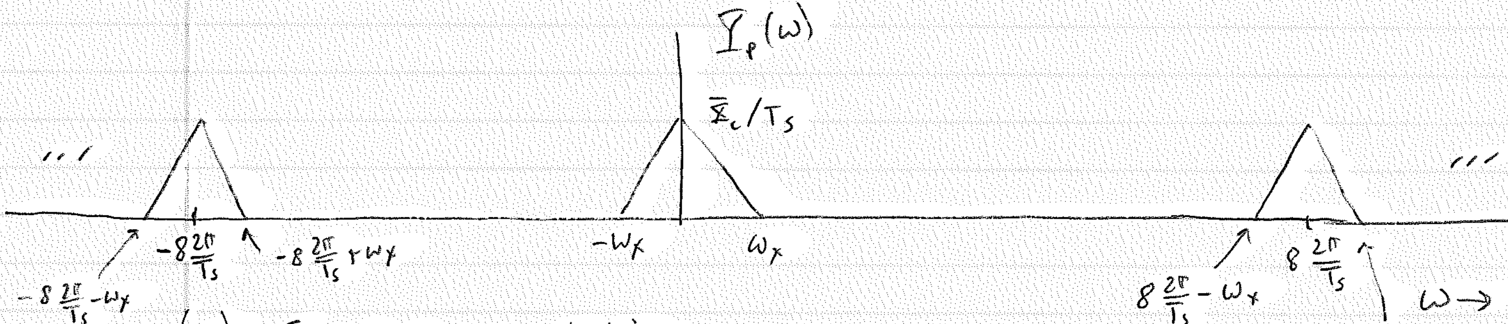
$$Y_p(\omega) = \sum_{n=-\infty}^{+\infty} y[n]\, \mathcal{F}\left\{\delta\left(t - n\frac{T_s}{M}\right)\right\} = \sum_{n=-\infty}^{+\infty} y[n]\, e^{-j\omega n \frac{T_s}{M}}\, 1$$

$$= \sum_{n=-\infty}^{+\infty} y[n]\, e^{-j\left(\omega\frac{T_s}{M}\right)n}$$

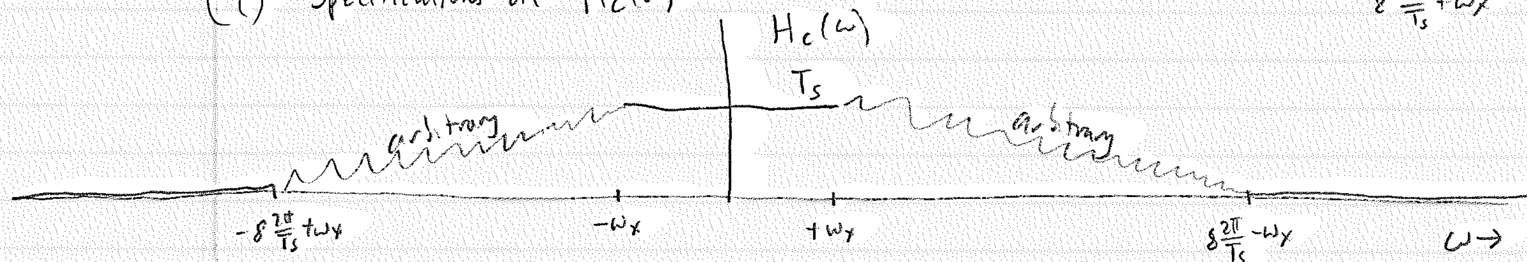
Compare $Y(\Omega)$ and $Y_p(\omega)$ to get that

$$Y_p(\omega) = Y\left(\Omega = \omega\frac{T_s}{M}\right)$$

$\frac{\omega T_s}{M} = 2\pi$ $\qquad$ $\frac{\omega T_s}{8} = \omega_x \frac{T_s}{8}$



(η) Specifications on $H_c(\omega)$:



(d) The reason this approach is used is that the specifications on $H_c(\omega)$, probably an analog circuit, are much less strict that if the oversampling had not been performed. The oversampling leads to a higher rate digital system and $H(\Omega)$ but this is not too burdensome for $1/T_s = 44.1\,kHz$ in a CD.

(e) The $H(\Omega)$ leads to an $h[n]$ that is a digital sinc. This function decays in both $n \to +\infty$ and $n \to -\infty$ directions but is non-zero for all n. It will have to be truncated in both directions. Then, if samples are retained to $n = -n_0$ $(n_0 > 0)$, a delay of $n_0 \frac{T_s}{M}$ will have to be added to the system in order to have a causal system. In many playback situations, such a delay can be tolerated. Much less delay can be tolerated in bi directional communication systems like voice telephony.