ECE 2200 / Spring 2016 / Problem Set 7 / Solution

P-9.17 1

$$y(t) = \int_{t-2}^{t+2} x(\tau)\,d\tau$$

(a) Compute the impulse response denoted by $h(t)$.

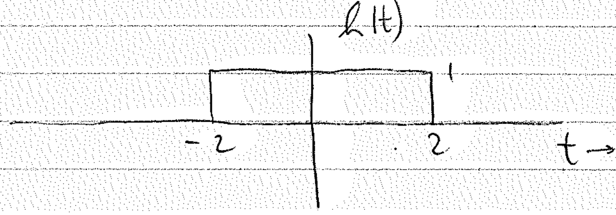
$x(t) = \delta(t)$.

$$y(t) = h(t) = \int_{t-2}^{t+2} \delta(\tau)\,d\tau$$

If the region of integration includes 0 then the value of the integral is $\int \delta(\tau)d\tau = 1$. Otherwise the value of the integral is 0.

$$= \begin{cases} 0 & t+2 < 0 \\ 1 & t+2 > 0 \text{ and } t-2 < 0 \\ 0 & t-2 > 0 \end{cases}$$

$$= \begin{cases} 0 & t < -2 \\ 1 & -2 < t < 2 \\ 0 & t > 2 \end{cases}$$



Not required (b) BIBO stable $\Longleftrightarrow \int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$.

$$\int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-2}^{2} dt = 4 < \infty$$

Therefore the system is BIBO stable.

Not required (c) Is the system causal? No since

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\,x(t-\tau)\,d\tau = \int_{-2}^{+2} x(t-\tau)\,d\tau$$

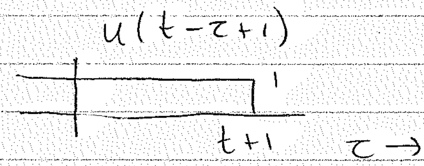
makes $y(t)$ dependent on $\{x(\xi) : \xi \in [-2, +2]\}$ which includes some of the future, specifically $\{x(\xi) : \xi \in (0, +2]\}$.

(d) if $x(t) = u(t+1)$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$$= \int_{-2}^{2} 1\; x(t-\tau)\,d\tau$$

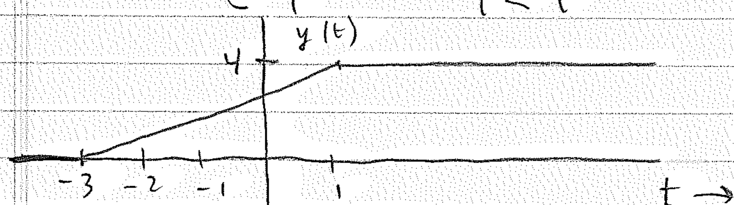
$$= \int_{-2}^{2} x(t-\tau)\,d\tau$$



$$= \int_{-2}^{2} u(t-\tau+1)\,d\tau$$

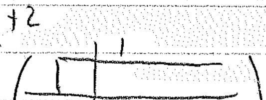
$$= \begin{cases} 0 & t+1 < -2 \\ \int_{-2}^{t+1} 1\,d\tau & -2 < t+1 < 2 \\ \int_{-2}^{2} 1\,d\tau & t+1 > 2 \end{cases}$$

$$= \begin{cases} 0 & t < -3 \\ t+1--2 & -3 < t < 1 \\ 4 & 1 < t \end{cases} = \begin{cases} 0 & t < -3 \\ t+3 & -3 < t < 1 \\ 4 & 1 < t \end{cases}$$



Alternative approach: $y(t) = \int_{t-2}^{t+2} x(\tau)\,d\tau = \int_{t-2}^{t+2} u(\tau+1)\,d\tau$

$$= \int_{t-2}^{t+2} \Big(\quad \Big)\,d\tau =$$



$$\begin{cases} 0 & t+2 < -1 \\ \int_{-1}^{t+2} 1\,d\tau & t-2 < -1 \text{ and } t+2 > -1 \\ \int_{t-2}^{t+2} 1\,d\tau & t-2 > -1 \end{cases}$$

$$= \begin{cases} 0 & t < -3 \\ t+2 - -1 & -3 < t < 1 \\ (t+2)-(t-2) & 1 < t \end{cases}$$

$$= \begin{cases} 0 & t < -3 \\ t+3 & -3 < t < 1 \\ 4 & 1 < t \end{cases}$$

= same answer!

$x[n] = \rho^{|n|} \qquad \rho = re^{j\phi} \qquad |\rho| < 1$

$$\bar{X}(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} \rho^{|n|} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{-1} \rho^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} \rho^{n} e^{-j\Omega n}$$

$m = -n$

$$= \sum_{m=1}^{\infty} \rho^{m} e^{-j\Omega(-m)} + \sum_{n=0}^{\infty} \left(\rho e^{-j\Omega}\right)^n$$

$$= \sum_{m=1}^{\infty} \left(\rho e^{j\Omega}\right)^m + \sum_{n=0}^{\infty} \left(\rho e^{-j\Omega}\right)^n$$

$$= \left[\sum_{m=0}^{\infty} \left(\rho e^{j\Omega}\right)^m\right] - 1 + \sum_{n=0}^{\infty} \left(\rho e^{-j\Omega}\right)^n$$

value of $(\rho e^{j\Omega})^m$ at $m = 0$

$$|\rho e^{j\Omega}| = |\rho||e^{j\Omega}| = |\rho| < 1 \qquad |\rho e^{-j\Omega}| = |\rho||e^{-j\Omega}| = |\rho| < 1$$

Therefore can use the formula $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$

on both sums to get

$$= \frac{1}{1-\rho e^{j\Omega}} - 1 + \frac{1}{1-\rho e^{-j\Omega}}$$

With ρ complex, I do not see a simple formula.

Need ① linearity of the discrete time Fourier transform and ② the time delay thm: $x[n] \leftrightarrow X(\Omega)$, $y[n] = x[n-n_0]$, then

$$Y(\Omega) = \sum_{n=-\infty}^{+\infty} y[n] e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x[n-n_0] e^{-j\Omega n} \overset{m=n-n_0}{=} \sum_{m=-\infty}^{+\infty} x[m] e^{-j\Omega(m+n_0)}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{+\infty} x[m] e^{-j\Omega m} = e^{-j\Omega n_0} X(\Omega).$$

(a) $y[n] = x[n] + 2x[n-1] + x[n-2]$. Linearity and delay thm imply

$$Y(\Omega) = X(\Omega) + 2e^{-j\Omega} X(\Omega) + e^{-j\Omega 2} X(\Omega) = X(\Omega)\left[1 + 2e^{-j\Omega} + e^{-j\Omega 2}\right].$$

Therefore, $H(\Omega) = \dfrac{Y(\Omega)}{X(\Omega)} = 1 + 2e^{-j\Omega} + e^{-j\Omega 2} = (1+e^{-j\Omega})^2$

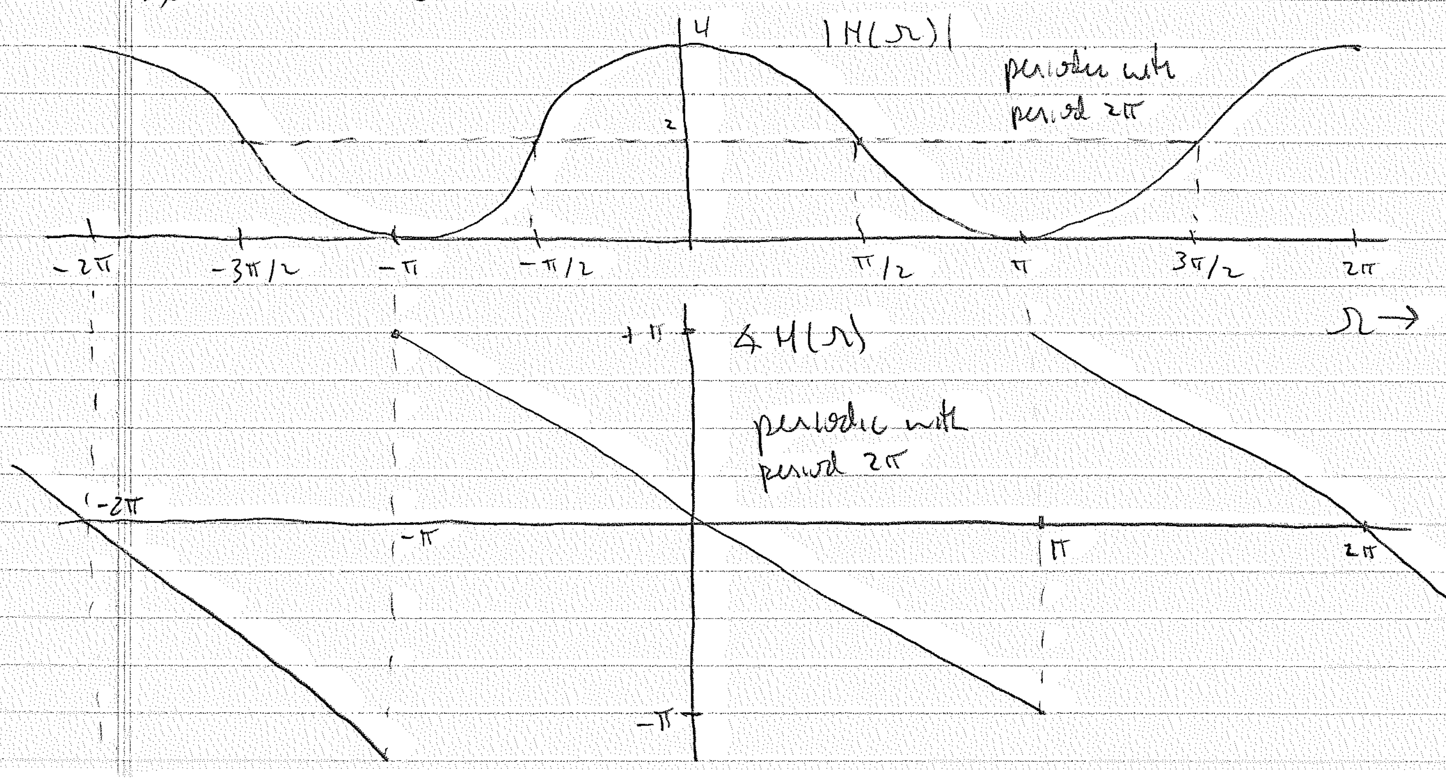
$$= \left[e^{-j\Omega/2}\left(e^{+j\Omega/2} + e^{-j\Omega/2}\right)\right]^2 = \left[e^{-j\Omega/2} 2\cos(\Omega/2)\right]^2$$

$$= e^{-j\Omega} 4\cos^2(\Omega/2).$$

(b) $|H(\Omega)| = 4\cos^2(\Omega/2) = 4\,\tfrac{1}{2}(1 + \cos(2\Omega/2))$

$$= 2(1 + \cos\Omega)$$

$\angle H(\Omega) = -\Omega$



(c) input $x[n] = 10 + 4\cos(0.5\pi n + \pi/4)$ (†)

$H(\Omega) = e^{-j\Omega} 4\cos^2(\Omega/2)$ satisfies $H(-\Omega) = H^*(\Omega)$.

Therefore, if the input is $x[n] = \cos(\Omega_0 n + \phi)$ then the output can be computed as follows:

$$x[n] = \cos(\Omega_0 n + \phi) = \tfrac{1}{2} e^{j\phi} e^{j\Omega_0 n} + \tfrac{1}{2} e^{-j\phi} e^{-j\Omega_0 n}$$

$$y[n] = \tfrac{1}{2} e^{j\phi} e^{j\Omega_0 n} H(\Omega_0) + \tfrac{1}{2} e^{-j\phi} e^{-j\Omega_0 n} H(-\Omega_0)$$

$$= \tfrac{1}{2} e^{j\phi} e^{j\Omega_0 n} H(\Omega_0) + \tfrac{1}{2} e^{-j\phi} e^{-j\Omega_0 n} H^*(\Omega_0)$$

$$= \tfrac{1}{2} e^{j\phi} e^{j\Omega_0 n} |H(\Omega_0)| e^{j\angle H(\Omega_0)} + \tfrac{1}{2} e^{-j\phi} e^{-j\Omega_0 n} |H(\Omega_0)| e^{-j\angle H(\Omega_0)}$$

$$= \tfrac{1}{2} e^{j\phi} e^{j\Omega_0 n} |H(\Omega_0)| e^{j\angle H(\Omega_0)} + \left(\tfrac{1}{2} e^{j\phi} e^{j\Omega_0 n} |H(\Omega_0)| e^{j\angle H(\Omega_0)}\right)^*$$

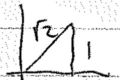
$$= 2\,\mathrm{Re}\left\{ \tfrac{1}{2} e^{j\phi} e^{j\Omega_0 n} |H(\Omega_0)| e^{j\angle H(\Omega_0)} \right\}$$

$$= |H(\Omega_0)| \cos(\Omega_0 n + \phi + \angle H(\Omega_0))$$

Apply this to the $x[n]$ in Eq (†) twice, once with $\Omega_0 = 0, \phi = 0$ and once with $\Omega_0 = 0.5\pi$, $\phi = \pi/4$ to get

$$y[n] = 10\,|H(0)| \cos(0n + 0 + \angle H(0))$$
$$+ 4\,|H(0.5\pi)| \cos(0.5\pi n + \pi/4 + \angle H(0.5\pi))$$

$|H(0)| = 4 \quad \angle H(0) = 0$



$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$|H(0.5\pi)| = 4\cos^2\left(\frac{\pi}{2}\,\frac{1}{2}\right) = 4\cos^2\left(\frac{\pi}{4}\right) = 4\left(\frac{1}{\sqrt{2}}\right)^2 = 2$$

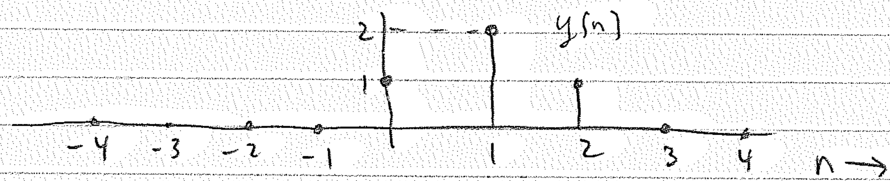
$\angle H(0.5\pi) = -\frac{\pi}{2}$

$$y[n] = 40 + 8\cos\left(0.5\pi n + \frac{\pi}{4} - \frac{\pi}{2}\right) = 40 + 8\cos\left(0.5\pi n - \frac{\pi}{4}\right)$$

(d) $x(n) = \delta(n)$. Easiest to do this in the time domain.

$$y(n) = x(n) + 2x(n-1) + x(n-2)$$

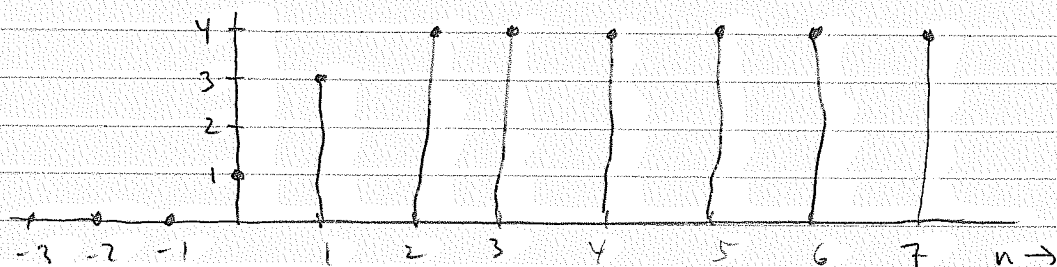
$$= \delta(n) + 2\delta(n-1) + \delta(n-2)$$



(e) $x(n) = u(n)$ Easiest to do this in the time domain.

$$y(n) = x(n) + 2x(n-1) + y(n-2)$$

$$= u(n) + 2u(n-1) + u(n-2)$$



note that $u[n] = \sum_{m=-\infty}^{n} \delta[m]$ so

$$(\text{answer to e})[n] = \sum_{m=-\infty}^{n} (\text{answer to d})[m] .$$

Because $x_c(t) \leftrightarrow X_c(\omega)$ satisfies the Nyquist sampling condition for this system we have shown that

$$Y_c(\omega) = \begin{cases} H(\Omega = \omega T_s)\, X_c(\omega) & |\omega| \le \frac{\pi}{T_s} = \frac{1}{2}\frac{2\pi}{T_s} = \frac{1}{2}\omega_s \\ 0 & \text{otherwise} \end{cases} \qquad (*)$$

So we need to compute $H(\Omega)$ from the definition

$$y[n] = \frac{1}{2M+1} \sum_{n=-M}^{+M} x[n-m]$$

$$Y(\Omega) = \frac{1}{2M+1} \sum_{m=-M}^{+M} e^{-j\Omega m}\, X(\Omega)$$

$$= X(\Omega) \frac{1}{2M+1} \sum_{m=-M}^{+M} e^{-j\Omega m}$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} \sum_{m=-M}^{+M} e^{-j\Omega m} e^{-j\Omega M}$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} \sum_{m=-M}^{+M} e^{-j\Omega(m+M)}$$

$$l = m + M$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} \sum_{l=0}^{2M} e^{-j\Omega l}$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} \sum_{l=0}^{2M} \left(e^{-j\Omega}\right)^{l}$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} \frac{1 - \left(e^{-j\Omega}\right)^{2M+1}}{1 - e^{-j\Omega}}$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} \frac{1 - e^{-j\Omega(2M+1)}}{1 - e^{-j\Omega}}$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} \frac{e^{-j\Omega(2M+1)/2}\left(e^{+j\Omega(2M+1)/2} - e^{-j\Omega(2M+1)/2}\right)}{e^{-j\Omega/2}\left(e^{+j\Omega/2} - e^{-j\Omega/2}\right)}$$

$$= X(\Omega) \frac{1}{2M+1} e^{+j\Omega M} e^{-j\Omega \frac{2M+1}{2}} e^{+j\Omega \frac{1}{2}} \frac{2j \sin[\Omega(2M+1)/2]}{2j \sin(\Omega/2)}$$

$$= X(\Omega) \frac{1}{2M+1} e^{j\Omega[M - (M+\frac{1}{2}) + \frac{1}{2}]} \frac{\sin[\Omega(2M+1)/2]}{\sin(\Omega/2)}$$

$$= X(\Omega) \frac{1}{2M+1} \frac{\sin[\Omega(2M+1)/2]}{\sin(\Omega/2)}$$

Therefore, $H(\Omega) = \frac{1}{2M+1} \frac{\sin \Omega(2M+1)/2}{\sin \Omega/2}$

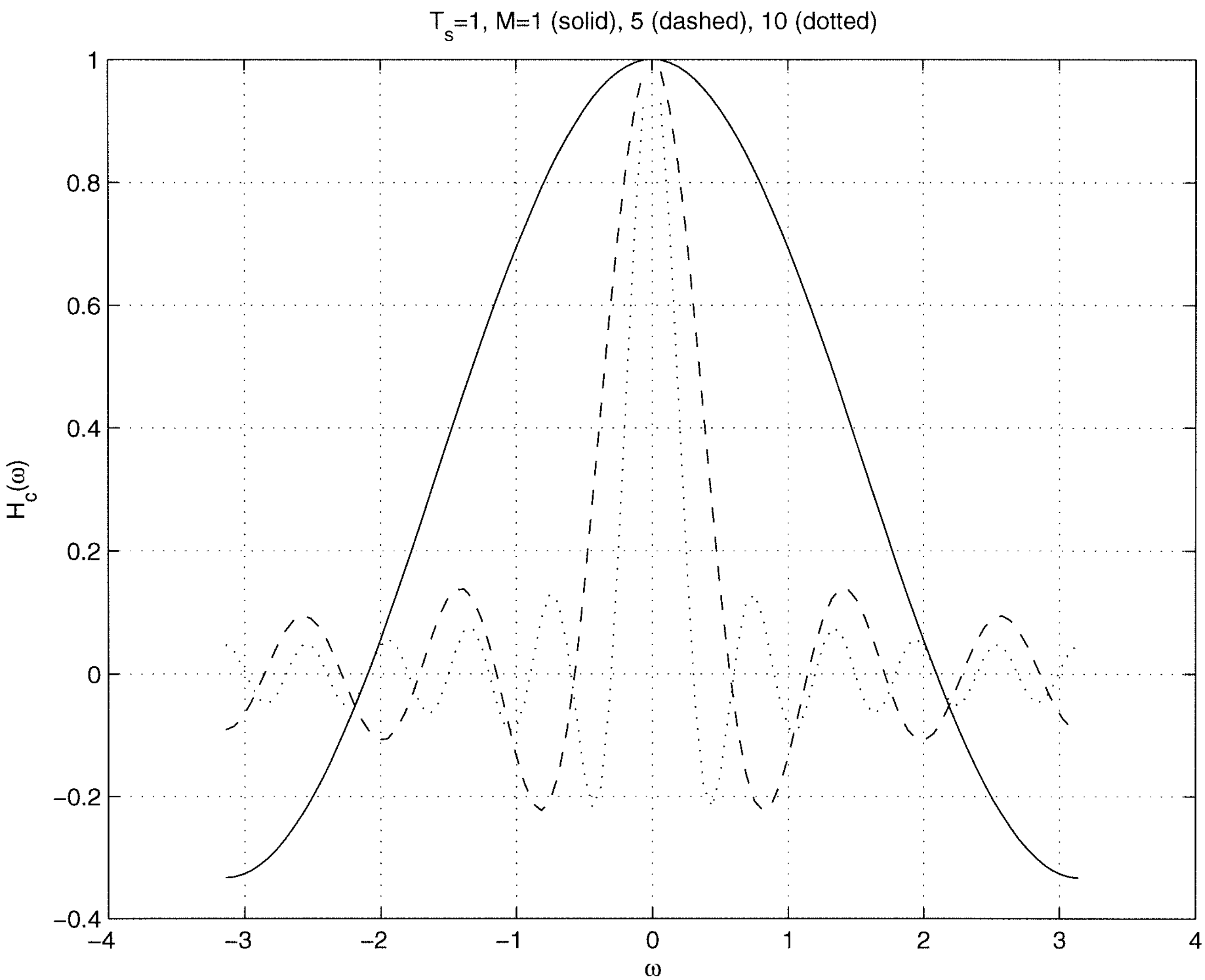
Therefore

$$H_c(\omega) = \frac{Y_c(\omega)}{X_c(\omega)} = H(\Omega = \omega T_s) \quad \text{for } |\omega| \le \tfrac{1}{2}\omega_s$$

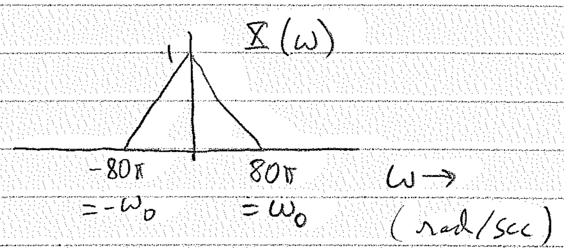
$$= \frac{1}{2M+1} \frac{\sin \omega T_s(2M+1)/2}{\sin \omega T_s/2}$$

Plot for $M \in \{1, 5, 10\}$. Plots on the next page are for $T_s = 1$, $\omega_s = \frac{2\pi}{T_s}$, and $\omega \in (-\omega_s/2, +\omega_s/2)$ since the system must be unaliased in order to work according to Eq (*).

p017 5


T_s=1, M=1 (solid), 5 (dashed), 10 (dotted)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
$H_c(\omega)$
−4
−3
−2
−1
0
1
2
3
4
ω

(a)



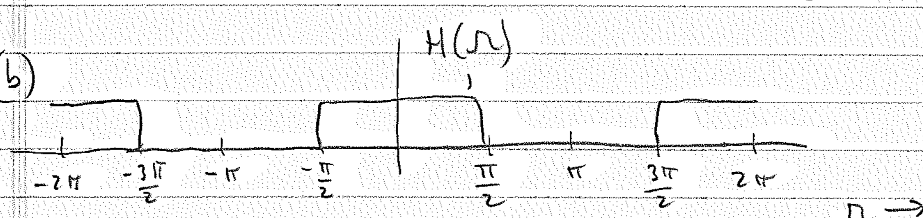
$X(\omega) = 0$ for $|\omega| \geq \omega_0$

Minimum sampling rate for which there is no aliasing is $\omega_s = 2\omega_0$

$= 160\pi$ rad/sec.

$$f_s = \frac{\omega_s}{2\pi} = \frac{160\pi}{2\pi} = 80 \ \frac{\text{cycles}}{\text{sec}} = 80\,\text{Hz}.$$

(b)



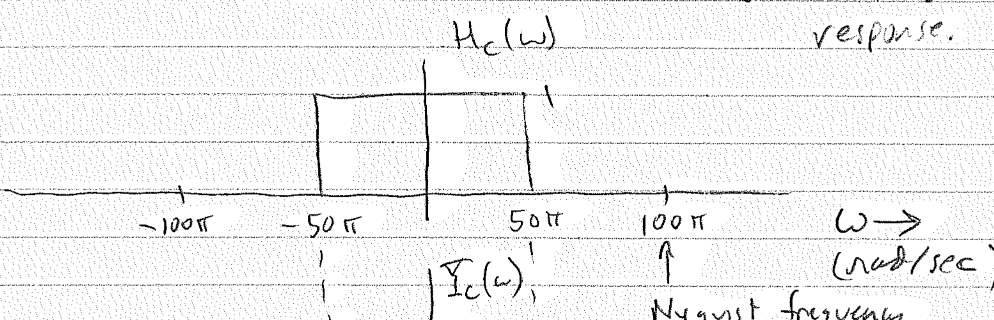
$f_s = 100$ Hz $\Longrightarrow T_s = \frac{1}{100}$ sec

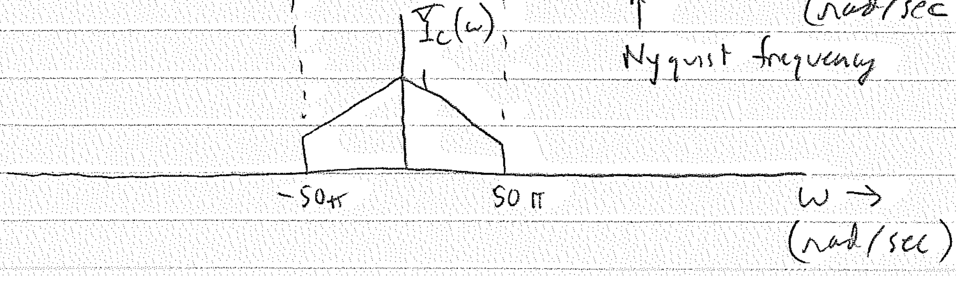
$$H_c(\omega) = \frac{Y_c(\omega)}{X_c(\omega)} = H(\Omega = \omega T_s) \quad \text{for} \quad |\omega| \leq \frac{\pi}{T_s}$$

(*) $\frac{\pi}{2} = \Omega = \omega T_s = \omega \frac{1}{100}$

$\Longrightarrow \omega = \frac{\pi}{2} 100 = 50\pi$

determine where the transition from 1 to 0 is located in the continuous time frequency response.





(c) Want to pick a new sampling rate such that the signal $X_c(\omega)$ is passed unaltered through the entire system

$\Longleftrightarrow$ Need to pick a new T_s so that the calculation in Part (b) at (*) gives an answer of $\omega = 80\pi$.

$$\frac{\pi}{2} = \Omega = \omega T_s = 80\pi T_s$$

$$\Longrightarrow T_s = \frac{\pi}{2} \frac{1}{80\pi} = \frac{1}{160} \text{ sec}$$

$$\Longrightarrow f_s = \frac{1}{T_s} = 160 \text{ Hz.}$$