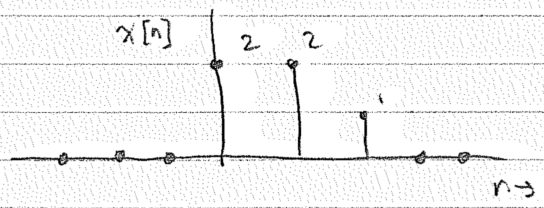
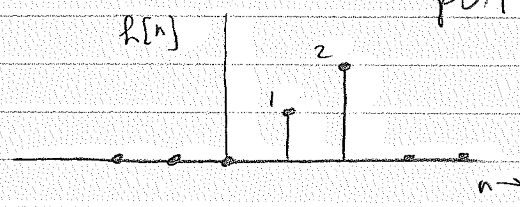
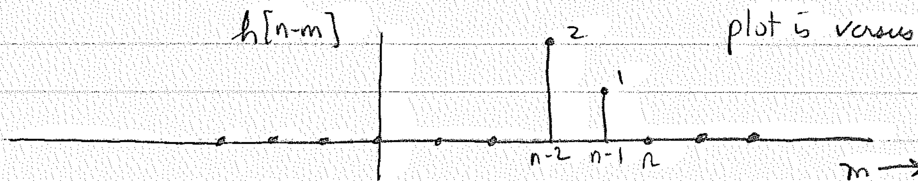




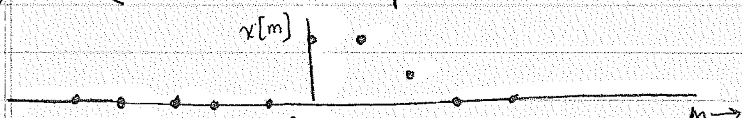
$$y[n] = x[n] * h[n]$$

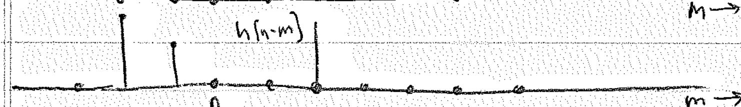
$$= \sum_{m=-\infty}^{+\infty} x[m] \underbrace{h[n-m]}_{h \text{ flipped and shifted}}$$

plot is versus m



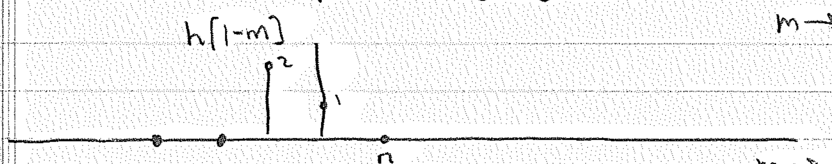
① if $n \leq 0$: no overlap





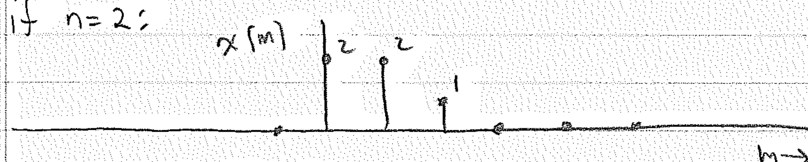
$$y[n] = \sum_{m=-\infty}^{+\infty} x[m] h[n-m] = \sum_{m=-\infty}^{+\infty} 0 = 0$$

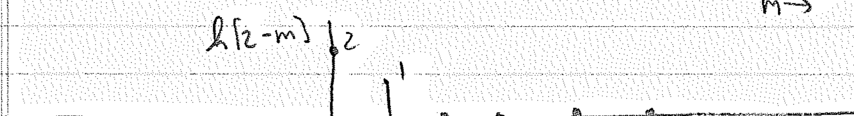
② if $n = 1$:



$$y[1] = \sum_{m=-\infty}^{+\infty} x[m] h[1-m] = (0)(2) + (2)(1) + (2)(0) + (1)(0) = 2$$

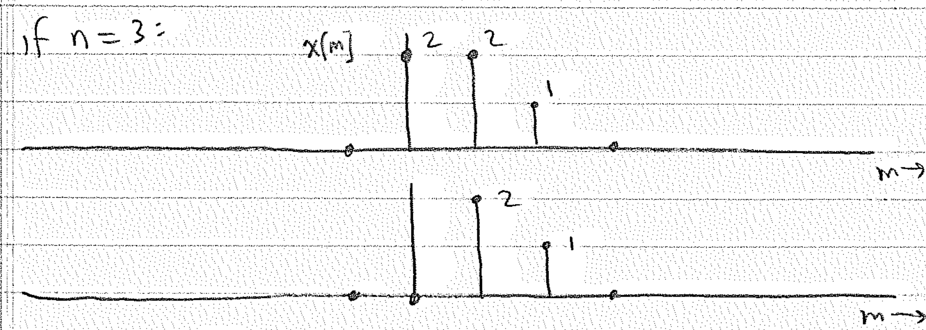
③ if $n = 2$:





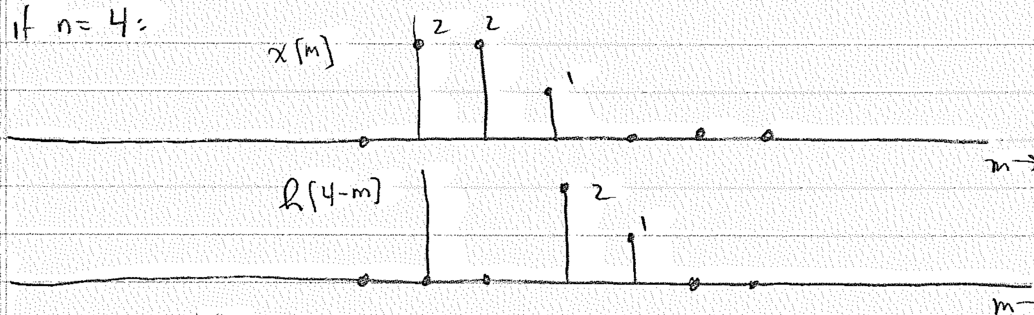
$$y[2] = \sum_{m=-\infty}^{+\infty} x[m] h[2-m] = (2)(2) + (2)(1) + (1)(0) = 6$$

(4) if $n = 3$:



$$y[3] = \sum_{m=-\infty}^{+\infty} x[m]\,h[3-m] = (2)(0) + (2)(2) + (1)(1) = 5$$

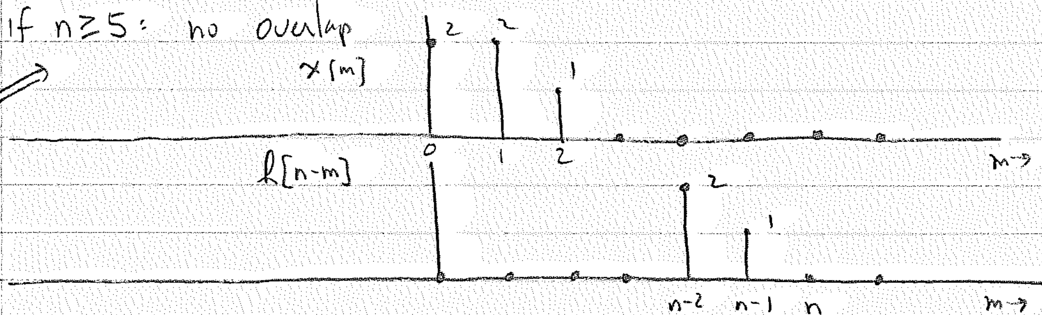
(5) if $n = 4$:



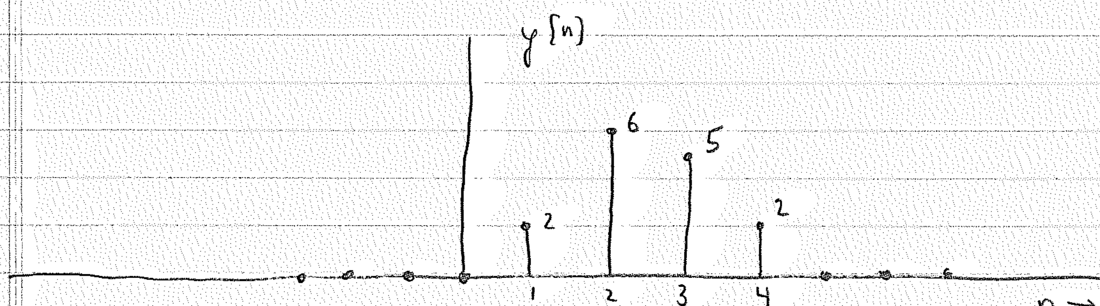
$$y[4] = \sum_{m=-\infty}^{+\infty} x[m]\,h[4-m] = (2)(0) + (2)(0) + (1)(2) + (0)(1) = 2$$

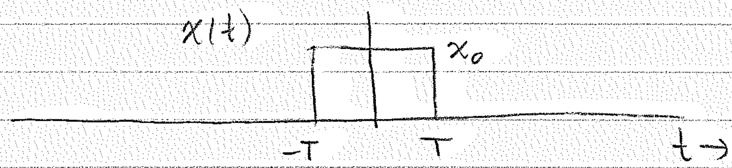
(6) if $n \geq 5$: no overlap

$n-2 \geq 3$

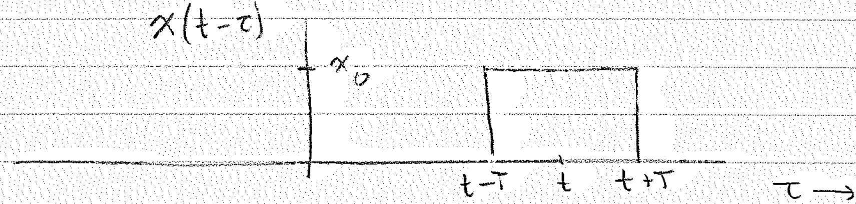


$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m] = \sum_{m=-\infty}^{+\infty} 0 = 0$$

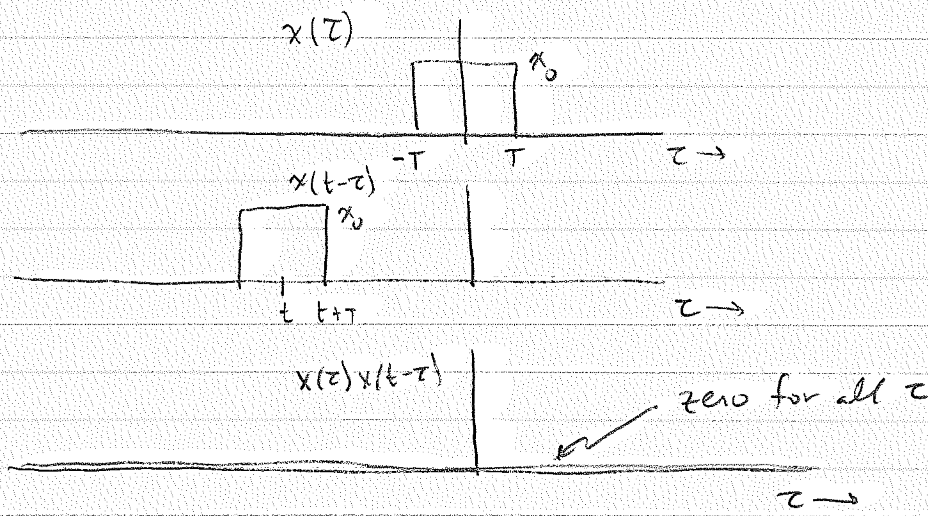




$$y(t) = x(t) * x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \underbrace{x(t-\tau)}_{x \text{ flipped and shifted}}\, d\tau$$

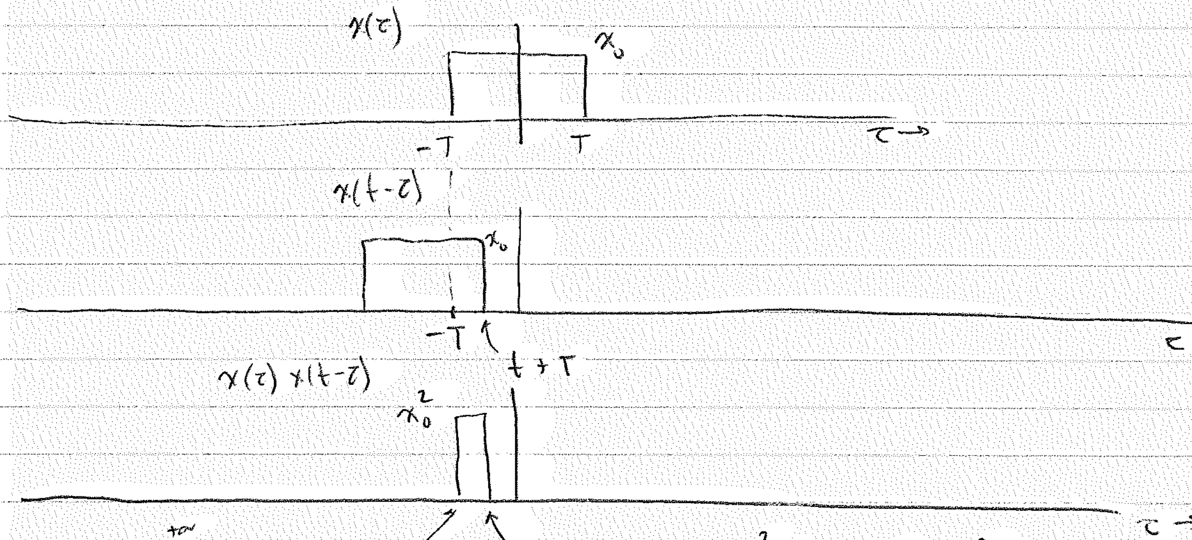


① if $t + T < -T \iff t < -2T$:



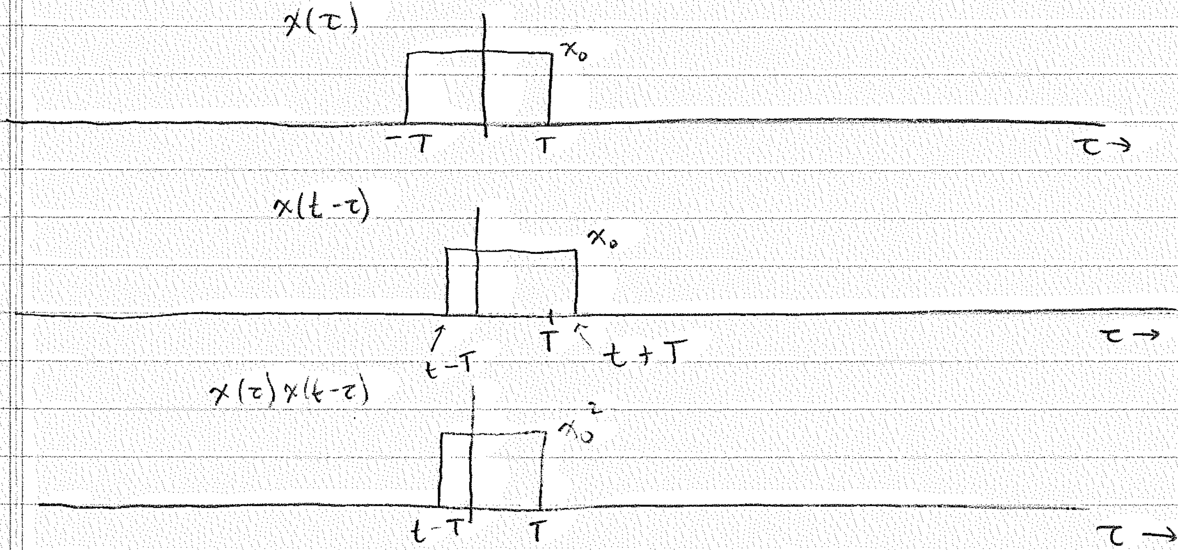
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} 0 \, d\tau = 0$$

② if $-T < t + T < T \iff -2T < t < 0$ "overlap trailing edge of $x(\tau)$"



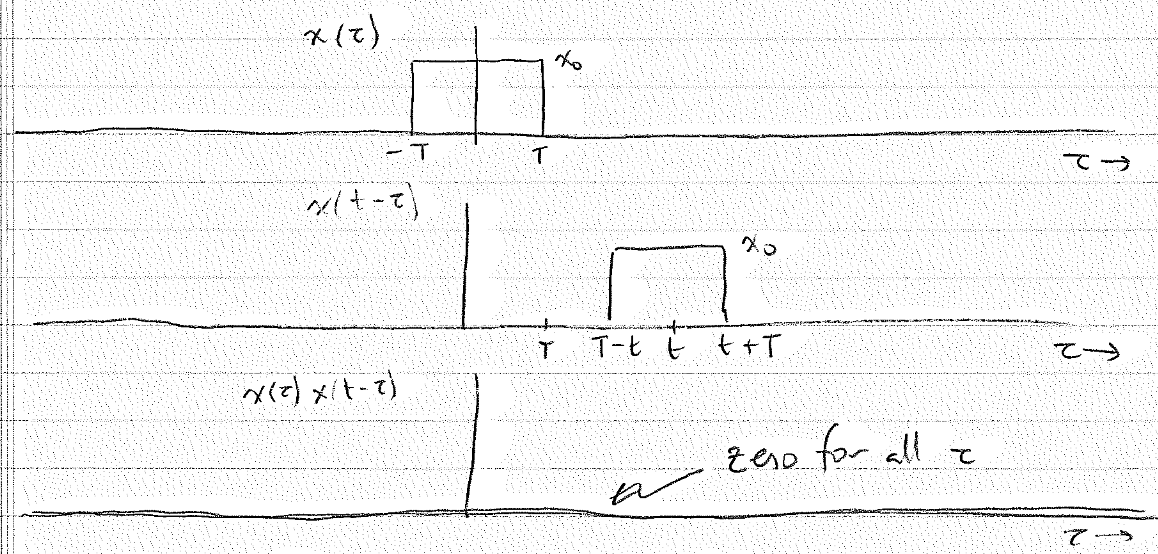
$$y(t) = \underbrace{\int_{-\infty}^{+\infty} x(\tau)\, x(t-\tau)\, d\tau}_{-T \quad t+T} = x_0^2(2T + t) = x_0^2(2T - |t|)$$

(3) if $T < t+T < 3T \Leftrightarrow 0 < t < 2T$ "overlap leading edge of $x(\tau)$"



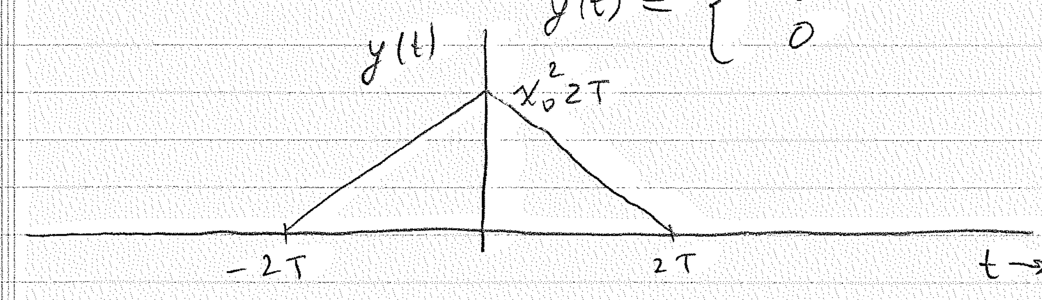
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)x(t-\tau)\,d\tau = x_0^2(2T-t) = x_0^2(2T-|t|)$$

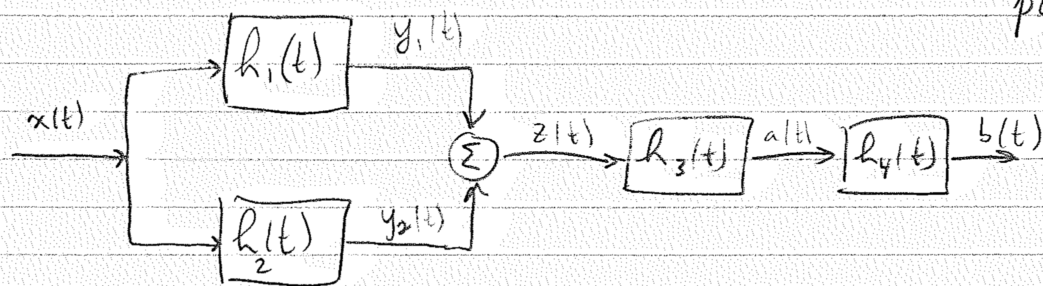
(4) if $t-T > T \Leftrightarrow t > 2T$



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} 0\,d\tau = 0$$

$$y(t) = \begin{cases} x_0^2(2T-|t|) & |t|<2T \\ 0 & \text{otherwise} \end{cases}$$





(a) $y_1(t) = x(t) * h_1(t)$

$y_2(t) = x(t) * h_2(t)$

$z(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t)$

convolution distributes over addition

$= x(t) * [h_1(t) + h_2(t)]$

$a(t) = z(t) * h_3(t) = \{x(t) * [h_1(t) + h_2(t)]\} * h_3(t)$

associative law

$= x(t) * \{[h_1(t) + h_2(t)] * h_3(t)\}$

$b(t) = a(t) * h_4(t) = [x(t) * \{[h_1(t) + h_2(t)] * h_3(t)\}] * h_4(t)$

$= x(t) * [\{[h_1(t) + h_2(t)] * h_3(t)\} * h_4(t)]$

but $b(t) = x(t) * h(t)$ so

$$h(t) = \{[h_1(t) + h_2(t)] * h_3(t)\} * h_4(t)$$

(b) if $h_4(t) * h_1(t) = \delta(t)$

$h(t) = \{[h_1(t) + h_2(t)] * h_3(t)\} * h_4(t)$

associative

$= [h_1(t) + h_2(t)] * \{h_3(t) * h_4(t)\}$

commutative

$= [h_1(t) + h_2(t)] * \{h_4(t) * h_3(t)\}$

associative

$= \{[h_1(t) + h_2(t)] * h_4(t)\} * h_3(t)$

convolution distributes over addition

$= (h_1(t) * h_4(t) + h_2(t) * h_4(t)) * h_3(t)$

commutative
$$= \left[h_4(t) * h_1(t) + h_2(t) * h_4(t) \right] * h_3(t)$$

$$= \left[\delta(t) + h_2(t) * h_4(t) \right] * h_3(t)$$

convolution distributes over addition
$$= \delta(t) * h_3(t) + \left(h_2(t) * h_4(t) \right) * h_3(t)$$

convolution with a delta function
$$= h_3(t) + \underbrace{\left(h_2(t) * h_4(t) \right) * h_3(t)}$$

it is equivalent to write these in any order and to put the parentheses around any pair

(a) $y(t) = \exp(x(t+2))$

Linear? No. A counter example to linear is the following:

$x_1(t) = 1 \Rightarrow y_1(t) = \exp(1)$

$x(t) = 2x_1(t) \Rightarrow y(t) = \exp(2) \neq 2y_1(t) = 2\exp(1)$.

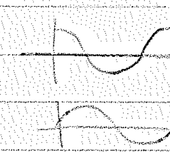
Time invariant? Yes.

$x_1(t) \Rightarrow y_1(t) = \exp(x_1(t+2))$

$x_2(t) = x_1(t-\tau) \Rightarrow y_2(t) = \exp(x_2(t+2)) = \exp(x_1(t+2-\tau))$

$= \exp(x_1([t-\tau]+2)) = y_1(t-\tau)$

(b) $y(t) = \cos(\omega_c t + x(t))$



Linear? No. A counter example is the following:

$x_1(t) = \pi$ for all t. $\Rightarrow y_1(t) = \cos(\omega_c t + \pi)$

$= \cos(\omega_c t)\underbrace{\cos(\pi)}_{-1} - \sin(\omega_c t)\underbrace{\sin(\pi)}_{0} = -\cos(\omega_c t)$

$x(t) = 2x_1(t) = 2\pi \Rightarrow y(t) = \cos(\omega_c t + 2\pi) = \cos(\omega_c t) \neq 2y_1(t)$

$= -2\cos(\omega_c t)$.

Time invariant? No. A counter example is the following:

$x_1(t) = -\omega_c t \Rightarrow y_1(t) = \cos(\omega_c t - \omega_c t) = \cos(0) = 1$

$x_2(t) = x_1(t-\tau) = -\omega_c(t-\tau) \Rightarrow y_2(t) = \cos(\omega_c t - \omega_c(t-\tau))$

$= \cos(\omega_c \tau) \neq 1$ if (for example) $\tau = \pi/\omega_c$

(c) $y(t) = [A + x(t)]\cos(\omega_c t)$

Linear? No. A counter example is the following:

$$x_1(t) \longrightarrow y_1(t) = [A + x_1(t)]\cos\omega_c t$$

$$x_2(t) \longrightarrow y_2(t) = [A + x_2(t)]\cos\omega_c t$$

$$x(t) = x_1(t) + x_2(t) \longrightarrow y(t) = [A + x_1(t) + x_2(t)]\cos\omega_c t$$

$$= [A + x_1(t)]\cos(\omega_c t) + [A + y_2(t)]\cos\omega_c t - A\cos\omega_c t$$

$$= y_1(t) + y_2(t) - A\cos\omega_c t \neq y_1(t) + y_2(t)$$

Time invariant? No. A counter example is the following:

$$x_1(t) = u(t) \longrightarrow y_1(t) = [A + u(t)]\cos(\omega_c t)$$

$$x_2(t) = x_1(t - \pi/\omega_c) = u(t - \pi/\omega_c) \longrightarrow y_2(t) = [A + u(t - \pi/\omega_c)]\cos(\omega_c t)$$

$$= [A + u(t - \pi/\omega_c)]\cos(\omega_c(t - \pi/\omega_c) + \pi)$$

$$= [A + u(t - \pi/\omega_c)]\left\{\cos(\omega_c(t - \pi/\omega_c))\underbrace{\cos\pi}_{=-1} - \sin(t - \pi/\omega_c)\underbrace{\sin\pi}_{=0}\right\}$$

$$= -[A + u(t - \pi/\omega_c)]\cos(\omega_c(t - \pi/\omega_c))$$

$$= -y_1(t - \pi/\omega_c) \neq y_1(t - \pi/\omega_c)$$

(d) $y(t) = \mathcal{E}ven\{x(t)\} = \dfrac{x(t) + x(-t)}{2}$

Linear? Yes, since

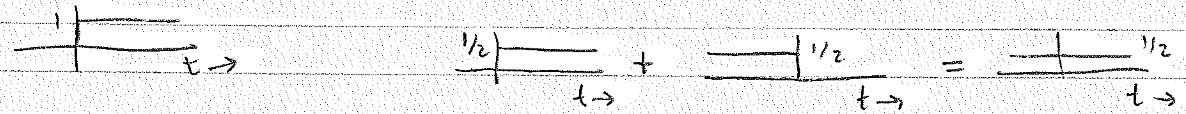
$$x_1(t) \to y_1(t) = \frac{x_1(t) + x_1(-t)}{2}, \quad x_2(t) \to y_2(t) = \frac{x_2(t) + x_2(-t)}{2}$$

implies $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow y(t) = \dfrac{\alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_1 x_1(-t) + \alpha_2 x_2(-t)}{2}$

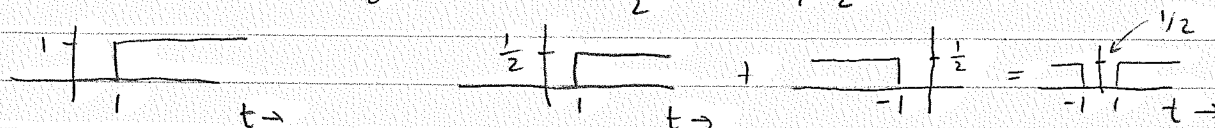
$$= \alpha_1\frac{x_1(t) + x_1(-t)}{2} + \alpha_2\frac{x_2(t) + x_2(-t)}{2} = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Time invariant? No. A counter example is the following:

$x_1(t) = u(t) \longrightarrow y_1(t) = \dfrac{u(t) + u(-t)}{2} = \frac{1}{2}$ for all t



$x_2(t) = u(t-1) \longrightarrow y_1(t) = \dfrac{u(t-1) + u(-t-1)}{2} \neq \frac{1}{2}$ for all t $= y_1(t-1)$



$$x(t) = e^{-\alpha t} u(t)$$

$$\begin{aligned}
y(t) &= x(t) * x(t) \\
&= \int_{-\infty}^{+\infty} x(\tau)\, x(t-\tau)\, d\tau \\
&= \int_{-\infty}^{+\infty} e^{-\alpha\tau} u(\tau)\, x(t-\tau)\, d\tau \\
&= \int_{0}^{\infty} e^{-\alpha\tau}\, x(t-\tau)\, d\tau \\
&= \int_{0}^{\infty} e^{-\alpha\tau} e^{-\alpha(t-\tau)} u(t-\tau)\, d\tau \\
&= u(t)\int_{0}^{t} e^{-\alpha\tau} e^{-\alpha(t-\tau)}\, d\tau \\
&= u(t)\int_{0}^{t} \underbrace{e^{-\alpha t}}_{\text{no dependence on } \tau}\, d\tau \\
&= u(t)\, e^{-\alpha t}\int_{0}^{t} d\tau \\
&= t\, e^{-\alpha t} u(t)
\end{aligned}$$

Exercise 9.4 p 265

$x(t) = e^{-at} u(t)$, $h(t) = e^{-bt} u(t)$, $y(t) = x(t) * h(t)$

$= \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t)$. Suppose $b = a$. Then

we get $y(t) = \frac{1}{a-a}\left(e^{-at} - e^{-at}\right)u(t) = \frac{0}{0} = ?$.

So use L'Hospital's Rule to evaluate $\lim_{b \to a} y(t)$.

$$\lim_{b \to a} y(t) = \lim_{b \to a} \frac{\frac{d}{db}\left[e^{-at} - e^{-bt}\right]}{\frac{d}{db}\left[b - a\right]} u(t)$$

$$= \lim_{b \to a} \frac{0 + t e^{-bt}}{1} u(t) = t e^{-at} u(t)$$

which is the same answer.