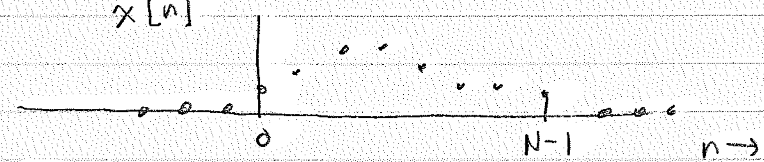
ECE 2200 / Spring 2016 / Problem Set 5 / Solutions

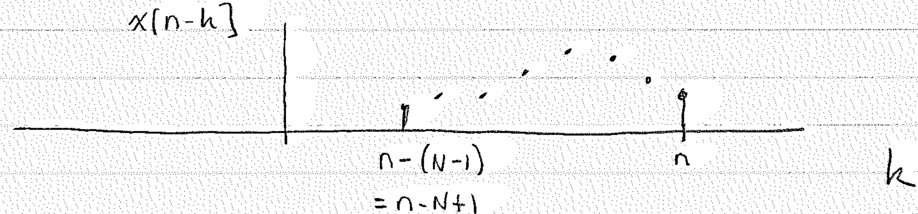
P-5.5 1

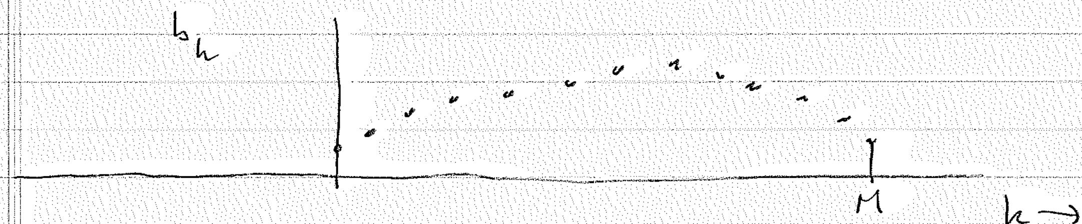
$$y[n] = \sum_{k=0}^{M} b_k \, x[n-k]$$

(a) $x[n] \neq 0$ only for $0 \le n \le N-1$



flip and shift: $x[n-k]$ versus k





Must have overlap between b_k and $x[n-k]$ in k.

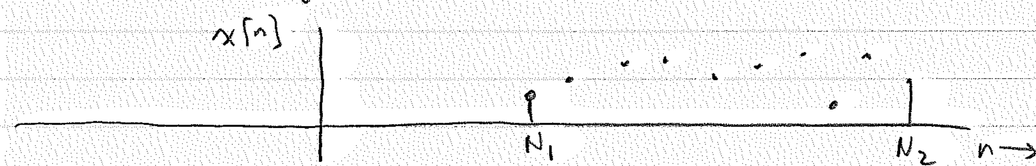
So answer is 0 (no overlap) if $n<0$ or $n-N+1 > M$

$$\Updownarrow$$

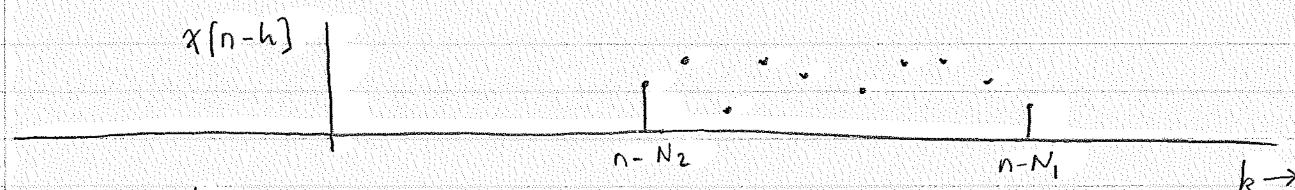
$$n > M+N-1$$

So answer may be non zero if $0 \le n \le \underbrace{M+N}_{P} - 1$. This is the support of $y[n]$

(b) $x[n] \neq 0$ only for $N_1 \le n \le N_2$ (so support of $x[n]$ is $n \in \{N_1, \ldots, N_2\}$)



flip and shift: $x[n-k]$ versus k



Same b_k plot as in Part (a).

Must have overlap between b_k and $x[n-k]$ in k.

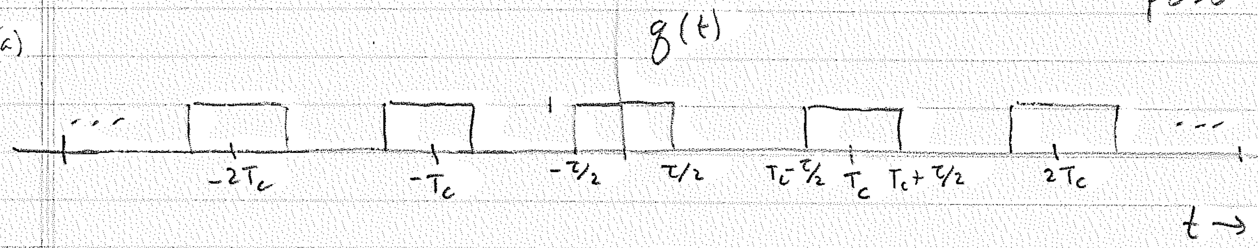
So answer is 0 (no overlap) if $n-N_1 < 0$ or $n - N_2 > M$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$n < N_1 \qquad\qquad n > M + N_2$$

So answer may be nonzero if $\underbrace{N_1}_{N_3} \le n \le \underbrace{M+N_2}_{N_4}$. This is the support of $y[n]$.

(a)



(b) a_k are the Fourier series coefficients of $g(t)$

$$a_k = \frac{1}{T_c}\int_{T_c} g(t)\, e^{-j\frac{2\pi}{T_c}kt}\, dt$$

$$= \frac{1}{T_c}\int_{-T_c/2}^{+T_c/2} g(t)\, e^{-j\frac{2\pi}{T_c}kt}\, dt$$

$$= \frac{1}{T_c}\int_{-T_c/2}^{+T_c/2} g_0(t)\, e^{-j\frac{2\pi}{T_c}kt}\, dt$$

$$= \frac{1}{T_c}\int_{-\infty}^{+\infty} g_0(t)\, e^{-j\frac{2\pi}{T_c}kt}\, dt$$

$$= \frac{1}{T_c}\left[\int_{-\infty}^{+\infty} g_0(t)\, e^{-j\omega t}\, dt\right]\Bigg|_{\omega=\frac{2\pi}{T_c}k}$$

$$= \frac{1}{T_c}\, Q_0\!\left(\omega = \frac{2\pi}{T_c}k\right)$$

(c) $$g(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{+j\frac{2\pi}{T_c}kt}$$

$$= \sum_{k=-\infty}^{+\infty} a_k\left[\int_{-\infty}^{+\infty} \delta\!\left(\omega - \frac{2\pi}{T_c}k\right) e^{+j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[2\pi\sum_{k=-\infty}^{+\infty} a_k\, \delta\!\left(\omega - \frac{2\pi}{T_c}k\right)\right] e^{+j\omega t}\, d\omega$$

Therefore $$Q(\omega) = 2\pi\sum_{k=-\infty}^{+\infty} a_k\, \delta\!\left(\omega - \frac{2\pi}{T_c}k\right)$$

$$= \frac{2\pi}{T_c}\sum_{k=-\infty}^{+\infty} Q_0\!\left(\omega = \frac{2\pi}{T_c}k\right)\delta\!\left(\omega - \frac{2\pi}{T_c}k\right)$$

(d) $a(t) = m(t)\,q(t)$

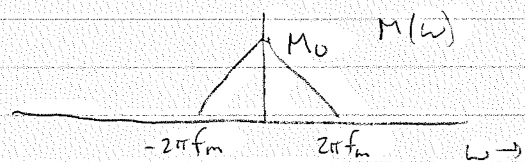
$\updownarrow$

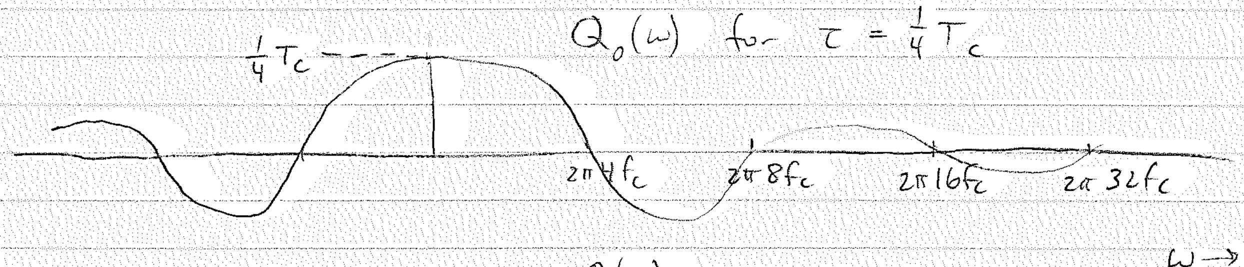
$$A(\omega) = \frac{1}{2\pi} M(\omega) * Q(\omega)$$

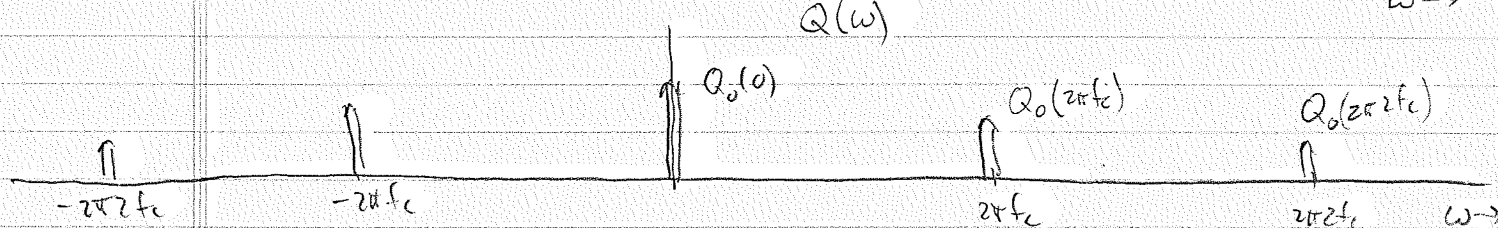
$$= \frac{1}{2\pi} M(\omega) * \frac{2\pi}{T_c} \sum_{h=-\infty}^{+\infty} Q_0\left(\omega = \frac{2\pi}{T_c} h\right) \delta\left(\omega - \frac{2\pi}{T_c} h\right)$$

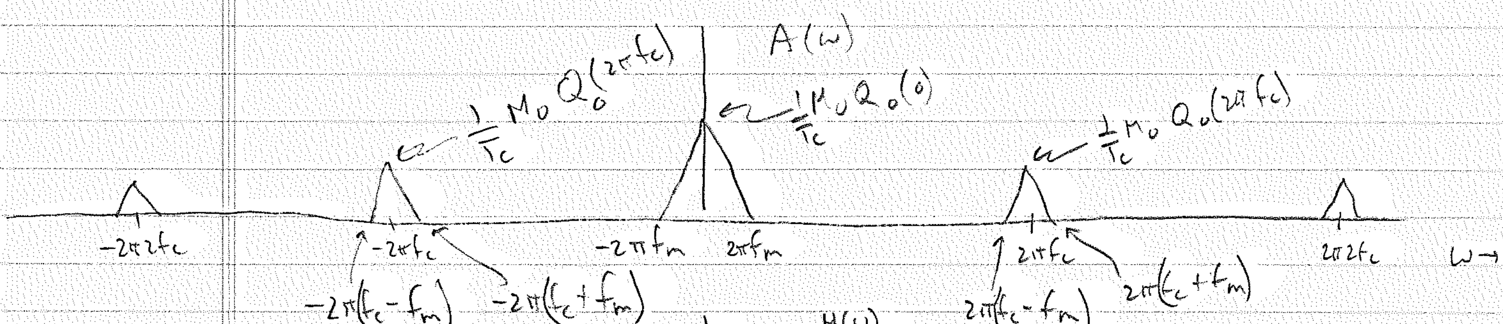
$$= \frac{1}{T_c} \sum_{h=-\infty}^{+\infty} Q_0\left(\omega = \frac{2\pi}{T_c} h\right) M\left(\omega - \frac{2\pi}{T_c} h\right)$$

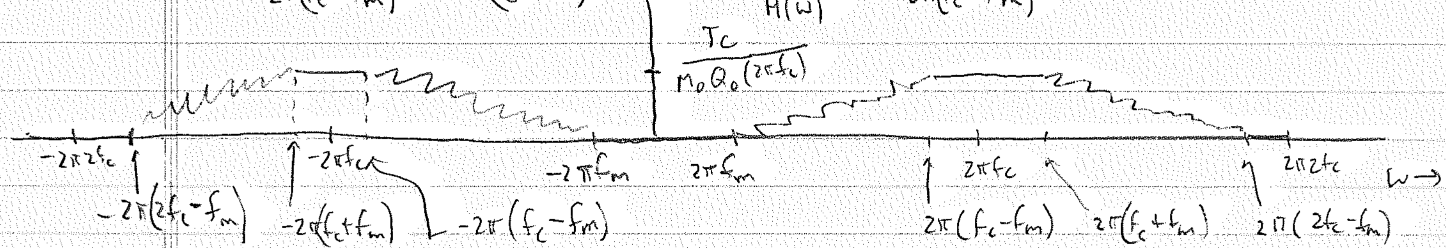
(e)











(f) $q^{new}(t) = q(t) - q_*$

$Q^{new}(\omega) = Q(\omega) - q_* 2\pi \delta(\omega)$

Need $Q^{new}(\omega = 0) = 0$

$$\Rightarrow \quad q_* = a_0$$

$$= \frac{1}{T_c} Q_0(0)$$

$$= \frac{1}{T_c} \tau \,\mathrm{sinc}(0)$$

$$= \frac{\tau}{T_c}$$

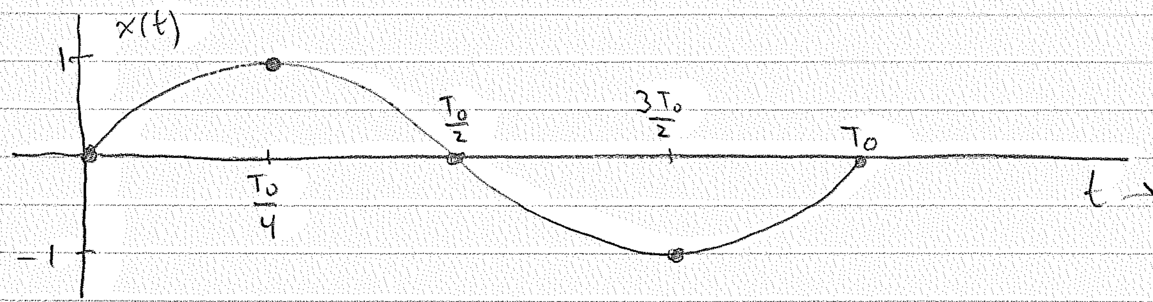
It is important to note that q_* changes the Fourier transform of $Q(\omega)$ only at the impulse at $\omega = 0$.

So the same value of H_0 will still work:

$$H_0 = \frac{T_c}{M_0 Q_0(2\pi f_c)}$$

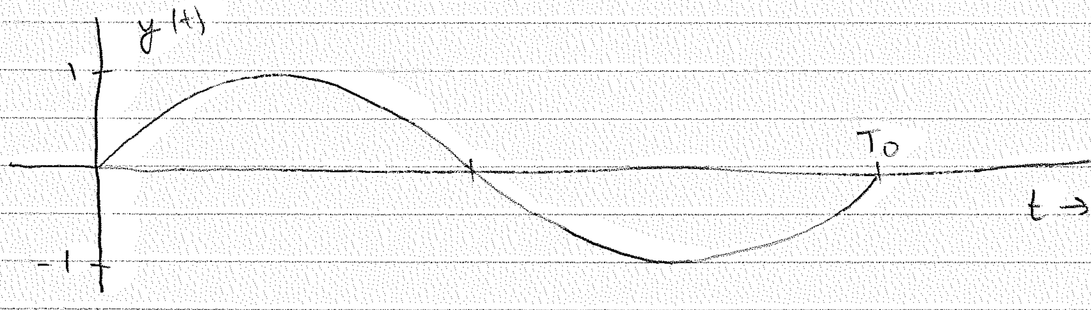
$x(t) = \sin \frac{2\pi}{T_0} t$. Highest frequency in $x(t)$ is $f_m = \frac{1}{T_0}$

(a) – (c) concern sampling at $T_s = \frac{1}{4} T_0$. The samples are:

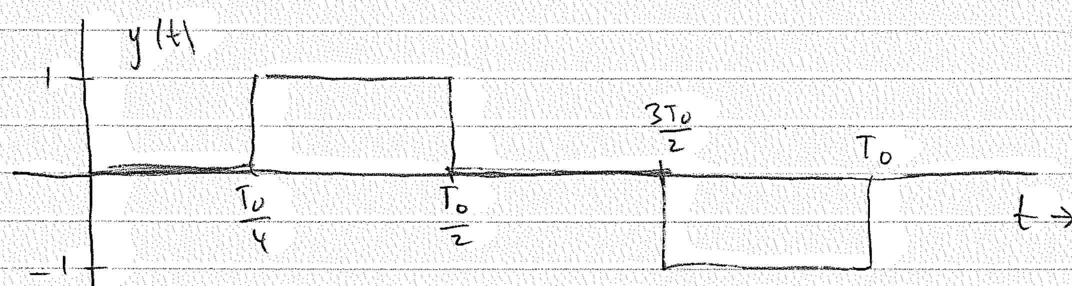


(a) No aliasing since $f_s = \frac{1}{T_s} = 4\frac{1}{T_0} > 2 f_m = 2\frac{1}{T_0}$

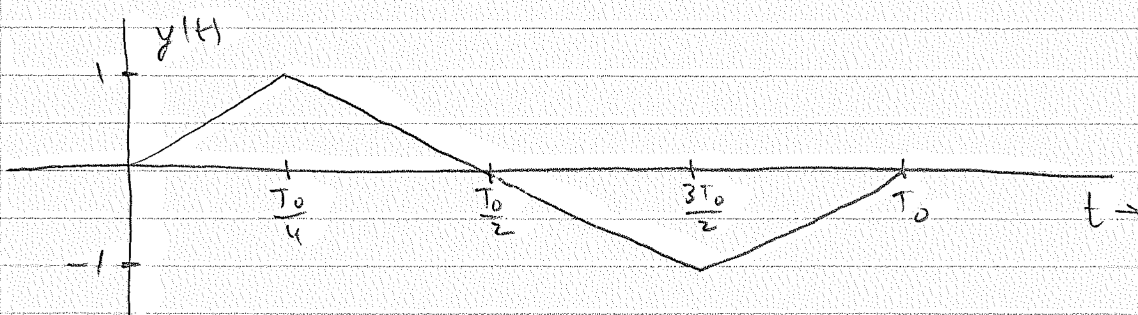
Therefore, $y(t) = x(t)$



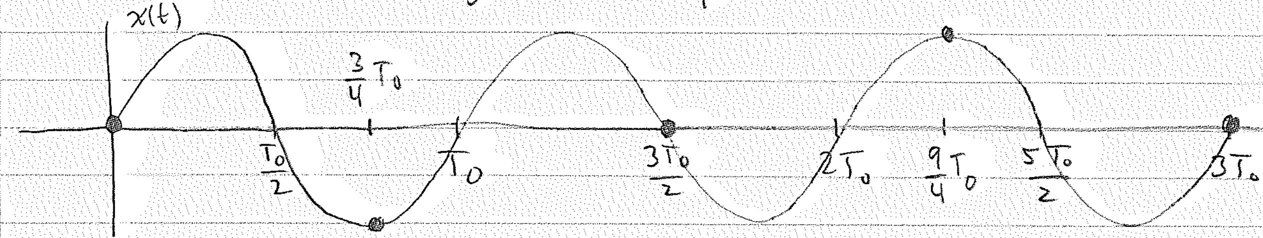
(b) sample and hold



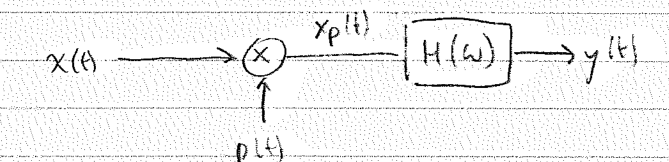
(c) linear interpolation



(d) – (f) concern sampling at $T_s = \frac{3}{4} T_0$. The samples are



(d) aliasing since $f_s = \frac{1}{T_s} = \frac{4}{3}\frac{1}{T_0} \not> 2f_m = 2\frac{1}{T_0}$. Compute the aliased frequency:



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \Leftrightarrow P(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right)$$

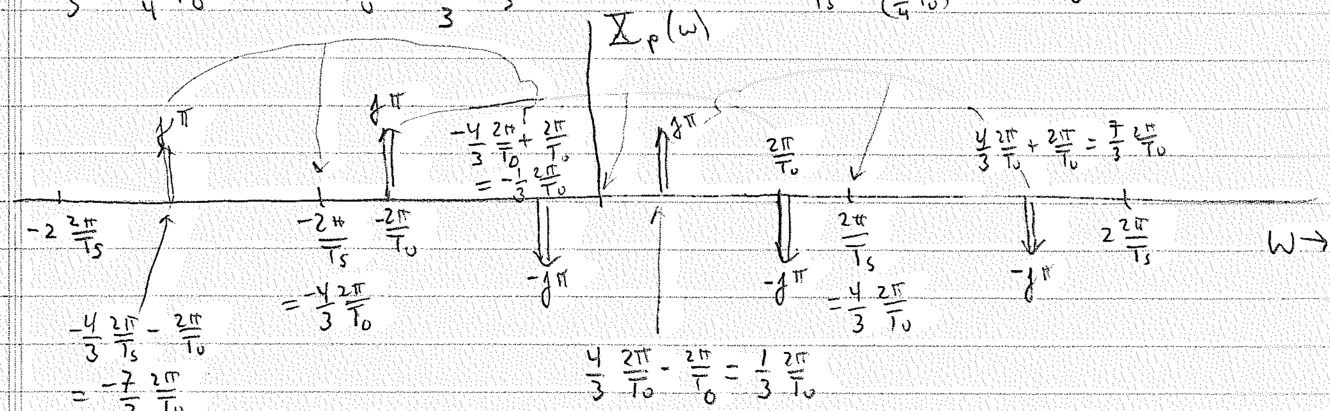
$$x_p(t) = x(t)p(t) \Leftrightarrow X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right)$$

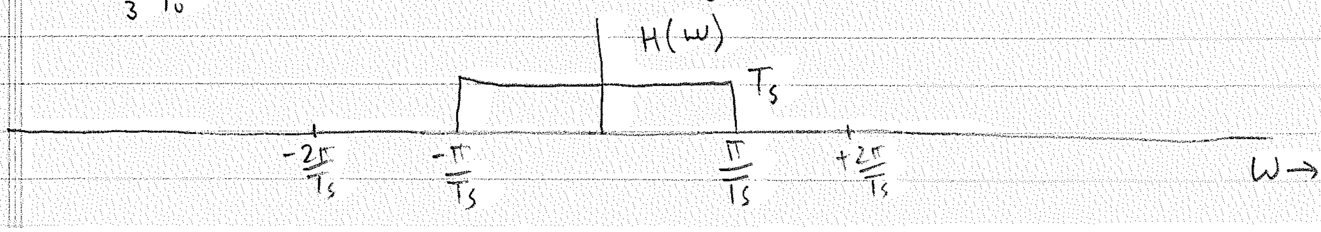
$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X\left(\omega - \frac{2\pi}{T_s} n\right)$$

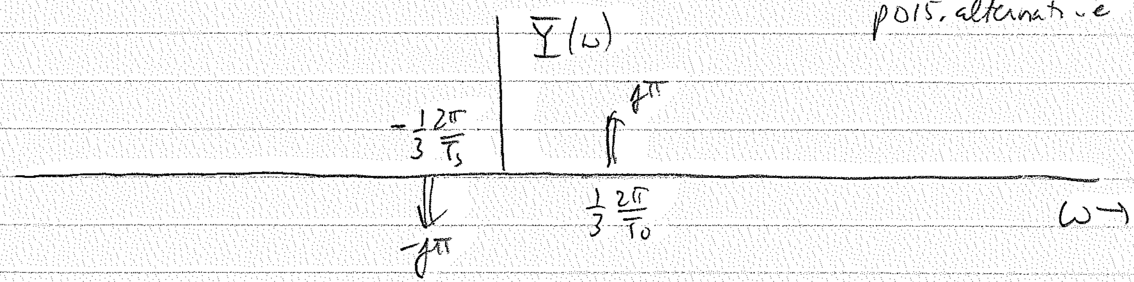
$$x(t) = \sin \frac{2\pi}{T_0} t \longleftrightarrow X(\omega) = 2\pi \frac{1}{2j} \left[\delta\left(\omega - \frac{2\pi}{T_0}\right) - \delta\left(\omega + \frac{2\pi}{T_0}\right) \right]$$

$$= -j\pi \left[\delta\left(\omega - \frac{2\pi}{T_0}\right) - \delta\left(\omega + \frac{2\pi}{T_0}\right) \right]$$

$$T_s = \frac{3}{4} T_0 \implies T_0 = \frac{4}{3} T_s \qquad \frac{2\pi}{T_s} = \frac{2\pi}{\left(\frac{3}{4} T_0\right)} = \frac{4}{3} \frac{2\pi}{T_0}$$



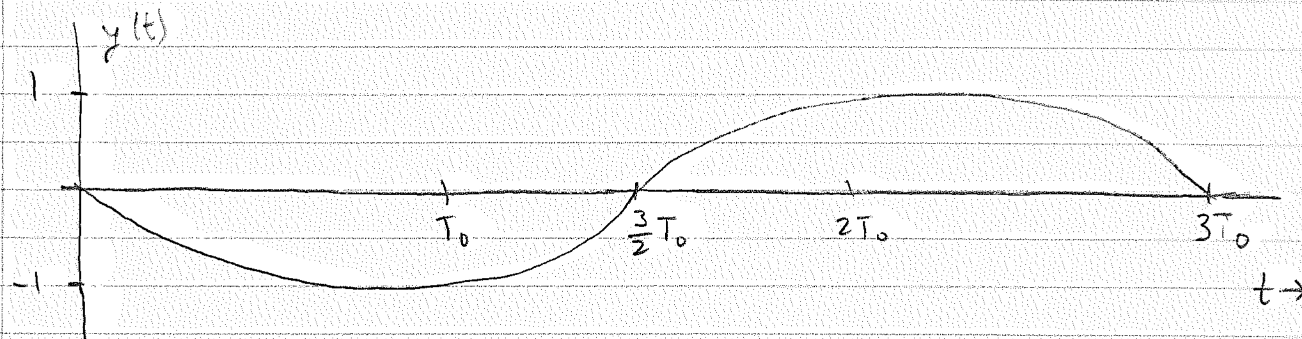




$$Y(\omega) = j\pi \left[\delta\left(\omega - \frac{1}{3}\frac{2\pi}{T_0}\right) - \delta\left(\omega + \frac{1}{3}\frac{2\pi}{T_0}\right) \right]$$

$$\updownarrow$$

$$y(t) = -\sin\left(\frac{1}{3}\frac{2\pi}{T_0}t\right)$$



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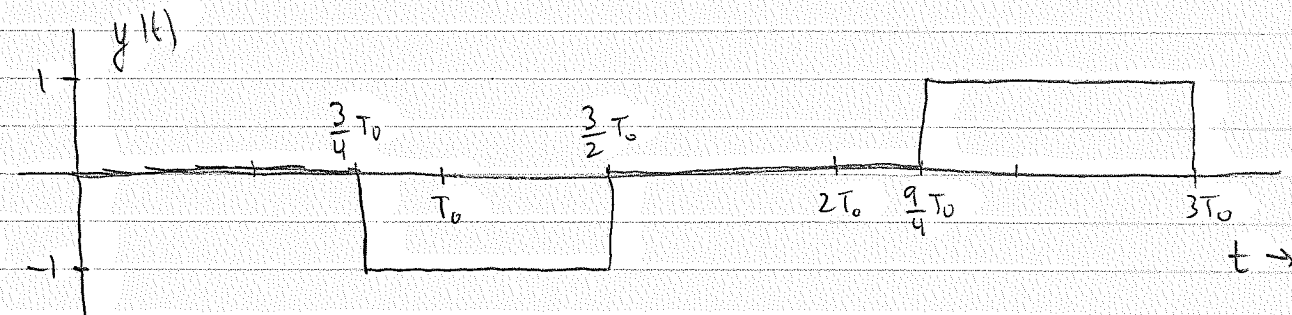
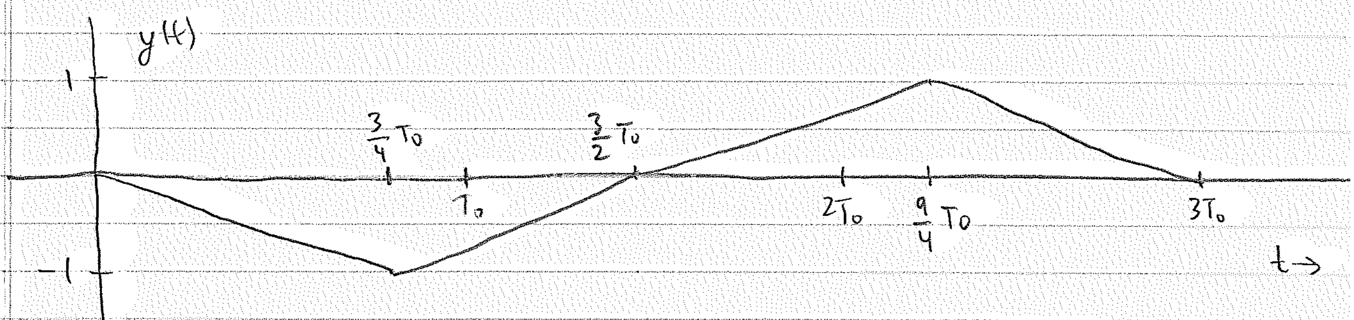


(f) linear interpolation



$$y[n] = \frac{1}{3}\left[x[n+1] + x[n] + x[n-1] \right]$$

$$x[n] = \delta[n]$$

$$\Rightarrow y[n] = \frac{1}{3}\left[\overbrace{\delta[n+1]}^{=1 \text{ for } n=-1} + \overbrace{\delta[n]}^{=1 \text{ for } n=0} + \underbrace{\delta[n-1]}_{=1 \text{ for } n=1} \right]$$

$$= \begin{cases} 0 & n < -1 \\ \frac{1}{3} & n = -1 \\ 1/3 & n = 0 \\ 1/3 & n = 1 \\ 0 & n > 1 \end{cases}$$

$$= \begin{cases} \frac{1}{3} & n \in \{-1, 0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

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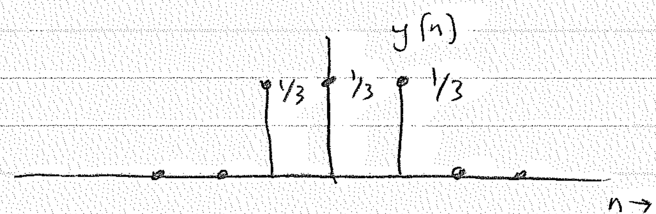

Usually call the output $h[n]$ when the input is $\delta[n]$