ECE 2200 / Spring 2016 / Problem set 4 / Solutions

P-4.8

$x(t) = \cos(2\pi f_0 t)\sin(2\pi f_1 t)$ $\qquad \sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$

$= \frac{1}{2}\sin(2\pi(f_0+f_1)t) + \frac{1}{2}\sin(2\pi(f_1-f_0)t)$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\omega)e^{+j\omega t}\,d\omega$$

So $y(t) = \sin\omega_0 t \longleftrightarrow Y(\omega) = \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$

Therefore

$$X(\omega) = \frac{1}{2}\frac{\pi}{j}\left[\delta(\omega - 2\pi(f_0+f_1)) - \delta(\omega + 2\pi(f_0+f_1))\right]$$
$$+ \frac{1}{2}\frac{\pi}{j}\left[\delta(\omega - 2\pi(f_1-f_0)) - \delta(\omega + 2\pi(f_1-f_0))\right]$$

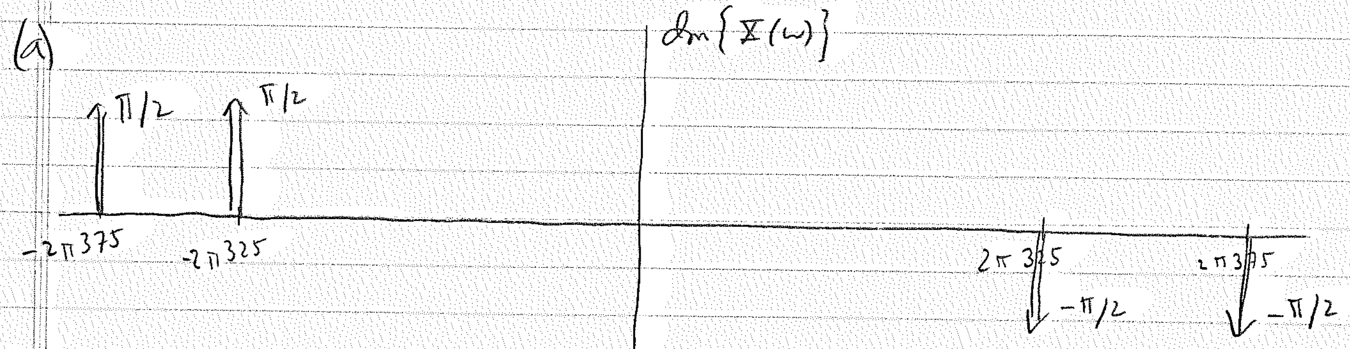
$$= -j\frac{\pi}{2}\left[\delta(\omega - 2\pi(f_1+f_0)) - \delta(\omega + 2\pi(f_1+f_0))\right]$$
$$-j\frac{\pi}{2}\left[\delta(\omega - 2\pi(f_1-f_0)) - \delta(\omega + 2\pi(f_1-f_0))\right]$$

So $\mathrm{Re}\, X(\omega) = 0$ and

$$\mathrm{Im}\{X(\omega)\} = -\frac{\pi}{2}\left[\delta(\omega - 2\pi(f_1+f_0)) - \delta(\omega + 2\pi(f_1+f_0))\right]$$
$$-\frac{\pi}{2}\left[\delta(\omega - 2\pi(f_1-f_0)) - \delta(\omega + 2\pi(f_1-f_0))\right]$$

$f_1 = 350\text{ Hz}$, $f_0 = 25\text{ Hz}$ $\qquad f_1 - f_0 = 325\text{ Hz}$, $f_1 + f_0 = 375\text{ Hz}$

(a)



(b) Nyquist sampling frequency = (2)(highest frequency in $X(\omega)$)

$= (2)(f_1+f_0) = 750\text{ Hz} = 2\pi 750\text{ rad/sec}$

$$\begin{tikzcd} x(t) \arrow[r] & \otimes \arrow[r, "x_r(t)"] & \boxed{H(\omega)} \arrow[r] & y(t) \\ & p(t) \arrow[u] & & \end{tikzcd}$$

$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$ period $= T_s$. Fourier Series coeffs are

$$c_n = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \left(\sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \right) e^{-j \frac{2\pi}{T_s} n t} \, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-j \frac{2\pi}{T_s} n t} \, dt$$

$$= \frac{1}{T_s}$$

Therefore

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{+j \frac{2\pi}{T_s} n t}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \left[\int_{-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right) e^{+j\omega t} \, d\omega \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right) \right] e^{+j\omega t} \, d\omega$$

$$\Longrightarrow P(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right)$$

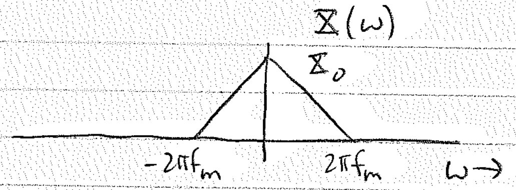
$$x_r(t) = x(t) p(t) \longleftrightarrow X_r(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

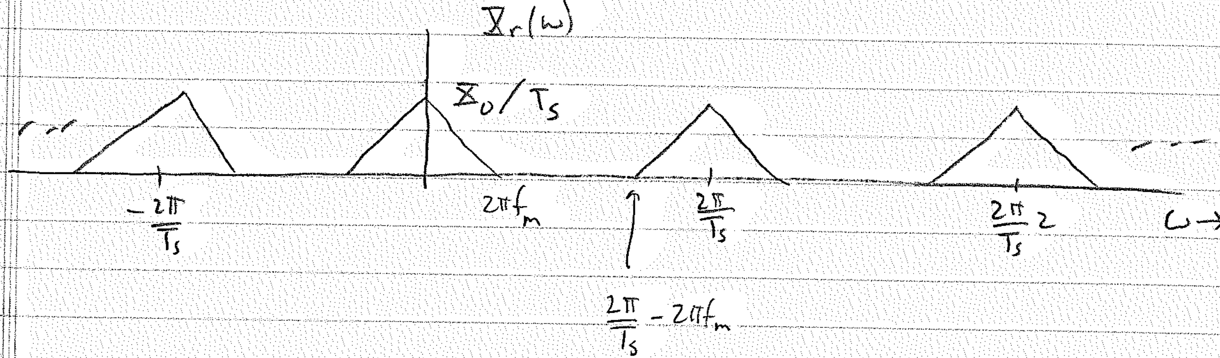
so that

$$X_r(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right)$$

$$= \frac{1}{2\pi} \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} X\left(\omega - \frac{2\pi}{T_s} n\right)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X\left(\omega - \frac{2\pi}{T_s} n\right)$$

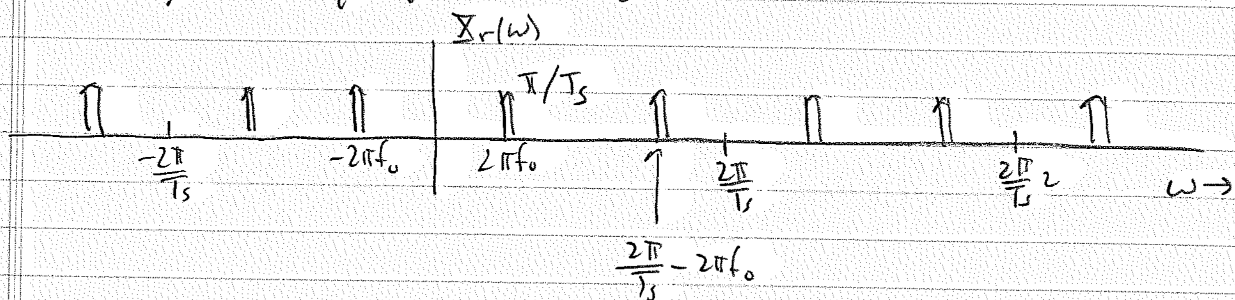




no aliasing if $2\pi f_m < \frac{2\pi}{T_s} - 2\pi f_m \iff 2 f_m < \frac{1}{T_s}$

$x(t) = \cos(2\pi f_0 t) \longleftrightarrow X(\omega) = \pi \left[\delta(\omega - 2\pi f_0) + \delta(\omega + 2\pi f_0) \right]$

So highest frequency in message is $2\pi f_0$

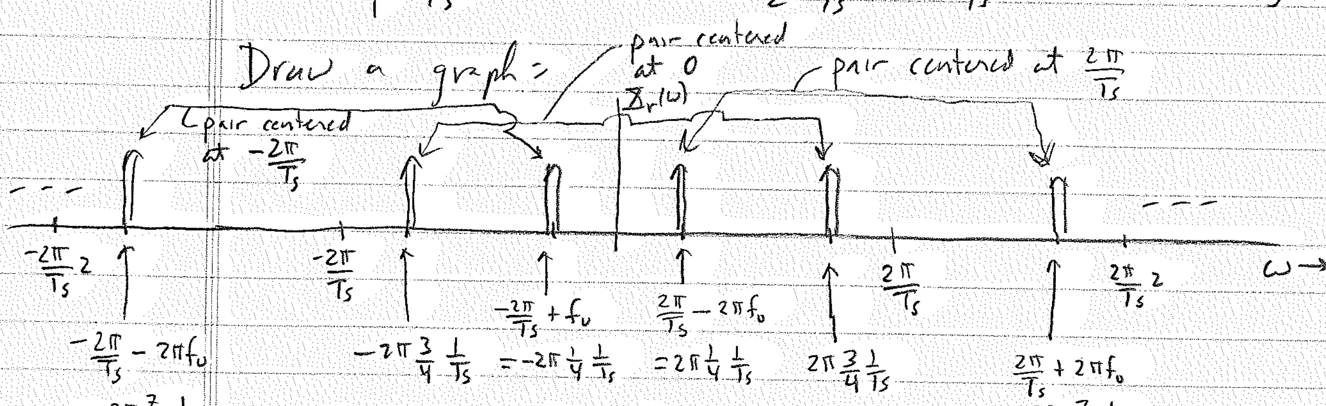


(a) $f_0 = \frac{1}{4} \frac{1}{T_s} \implies 2 f_0 = \frac{1}{2} \frac{1}{T_s} < \frac{1}{T_s} \implies$ no aliasing,

$y(t) = x(t)$

(b) $f_0 = \frac{3}{4} \frac{1}{T_s} \implies 2 f_0 = \frac{3}{2} \frac{1}{T_s} > \frac{1}{T_s} \implies$ aliasing.

Draw a graph:



Only the pair of impulses at $\pm 2\pi \frac{1}{4} \frac{1}{T_s}$ are passed by the filter.

Therefore the output is

$$y(t) = \cos\left(2\pi \frac{1}{4} \frac{1}{T_s}\right)$$

not

$$x(t) = \cos\left(2\pi \frac{3}{4} \frac{1}{T_s}\right)$$

(c) $x(t) = \cos(2\pi f_0 t)$, $y(t) = \cos(2\pi f_1 t)$

Want two values of f_0 such that $f_1 = \frac{1}{8} \frac{1}{T_s}$

Note that f_1 is within the pass band of the filter so this is possible.

One value of f_0 is the unaliased situation: $\boxed{f_0 = \frac{1}{8} \frac{1}{T_s}}$.

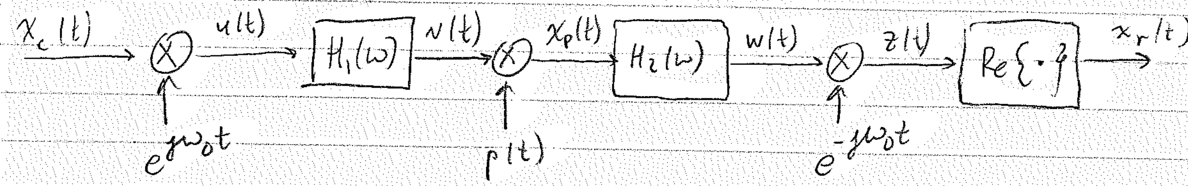
The next smallest value of f_0 is the case we worked out in Part (b) where the impulses that are in the pass band of the filter are one of the pair centered at $\frac{1}{T_s}$ and one of the pair centered at $\frac{-1}{T_s}$.

The equation is

$$\frac{2\pi}{T_s} - 2\pi f_0 = f_1 = 2\pi \frac{1}{8} \frac{1}{T_s}$$

$$\Longleftrightarrow \quad \frac{1}{T_s} - f_0 = \frac{1}{8} \frac{1}{T_s}$$

$$\Longleftrightarrow \quad \boxed{f_0 = \frac{7}{8} \frac{1}{T_s}}$$



(a) $x_c(t) \in \mathbb{R}$

(b) $X_c(\omega) = 0$ except for $\omega_1 \leq |\omega| \leq \omega_2$

(c) $\omega_0 = (\omega_1 + \omega_2)/2$

(d) $H_1(\omega) = \begin{cases} 1 & |\omega| \leq (\omega_2 - \omega_1)/2 \\ 0 & \text{otherwise} \end{cases}$

(e) $p(t) = \sum_{n=-\infty}^{+\infty} \delta\left(t - n\frac{2\pi}{\omega_s}\right) \quad \left(\frac{2\pi}{\omega_s} = T_s\right) \Leftrightarrow P(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta(\omega - \omega_s n)$

(f) $\omega_s = 2(\omega_2 - \omega_1)$

(g) $H_2(\omega) = \begin{cases} 2T_s & |\omega| \leq (\omega_2 - \omega_1)/2 \\ 0 & \text{otherwise} \end{cases}$

(h) $x_r(t) = Re\{z(t)\} = \frac{1}{2}[z(t) + z^*(t)]$

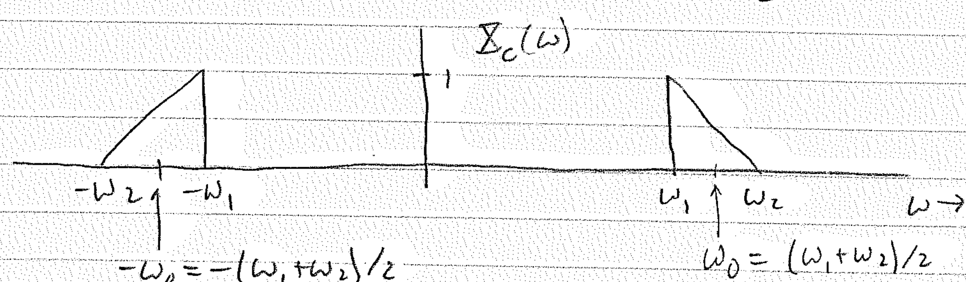
$\Updownarrow$

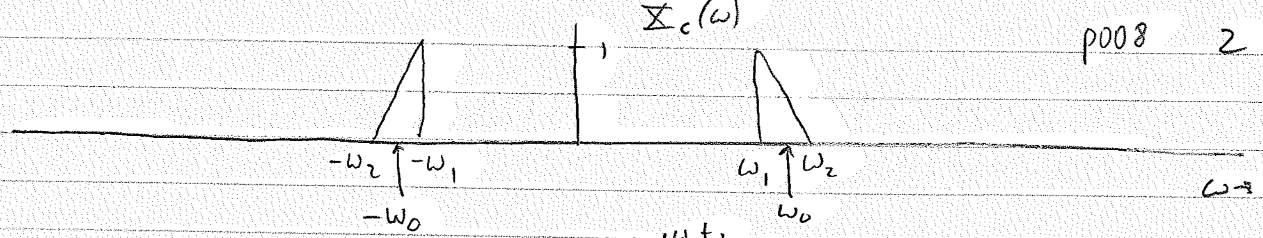
$X_r(\omega) = \frac{1}{2}\left(Z(\omega) + \mathcal{F}\{z^*(t)\}\right)$

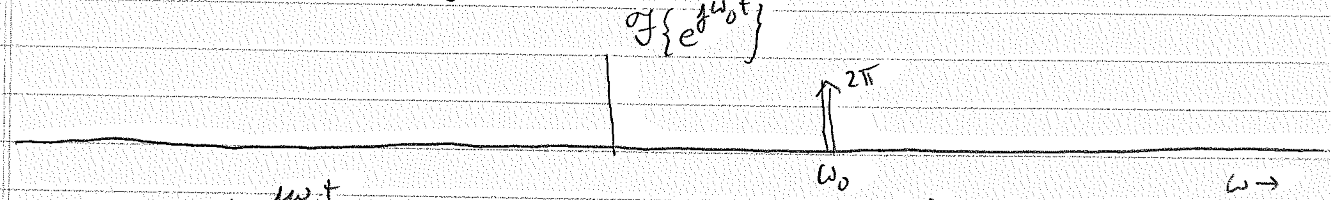
$$\mathcal{F}[z^*(t)] = \int_{-\infty}^{+\infty} z^*(t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \left[z(t) e^{-j(-\omega)t}\right]^* dt$$

$$= \left[\int_{-\infty}^{+\infty} z(t) e^{-j(-\omega)t} dt\right]^* = Z^*(-\omega)$$

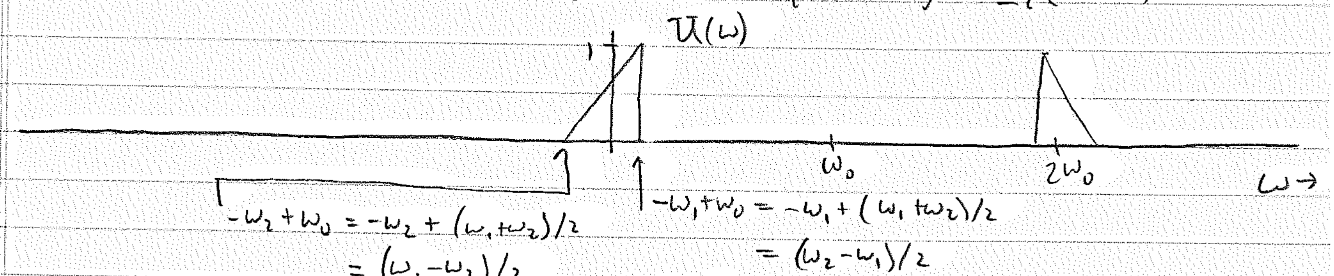
So $X_r(\omega) = \frac{1}{2}\left[Z(\omega) + Z^*(-\omega)\right]$

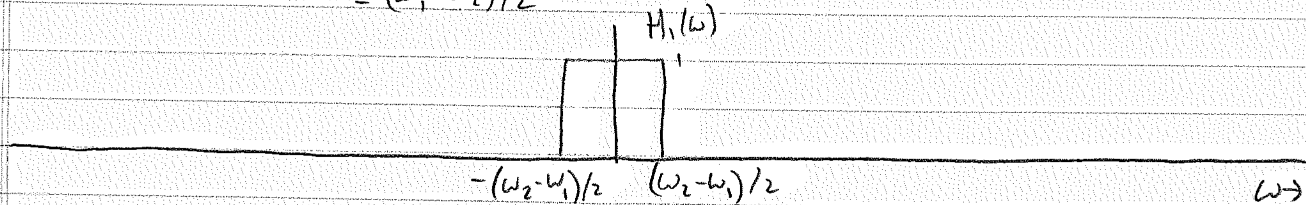


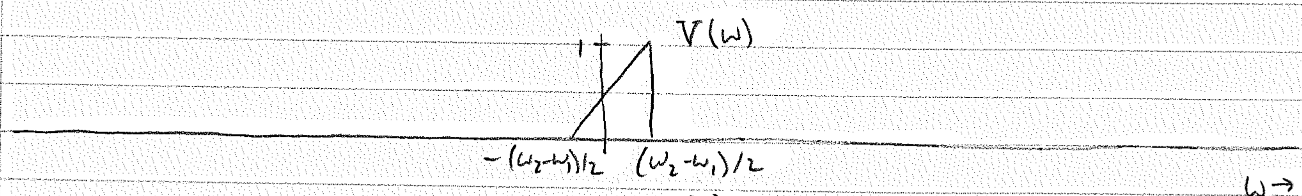


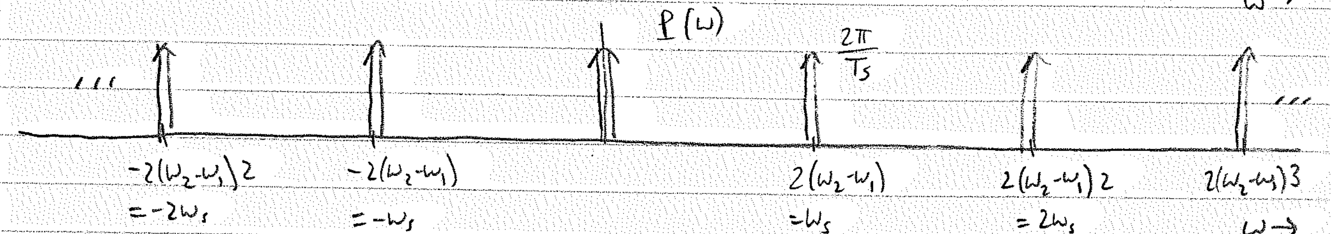


$$u(t) = x_c(t)\, e^{j\omega_0 t} \leftrightarrow U(\omega) = \frac{1}{2\pi} X_c(\omega) * \mathcal{F}\{e^{j\omega_0 t}\} = X_c(\omega - \omega_0)$$



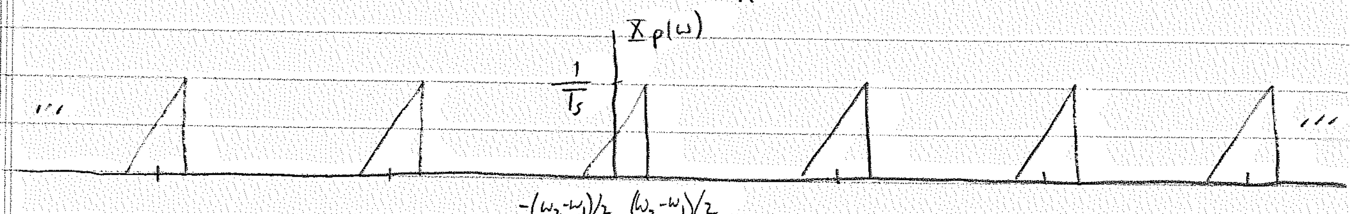


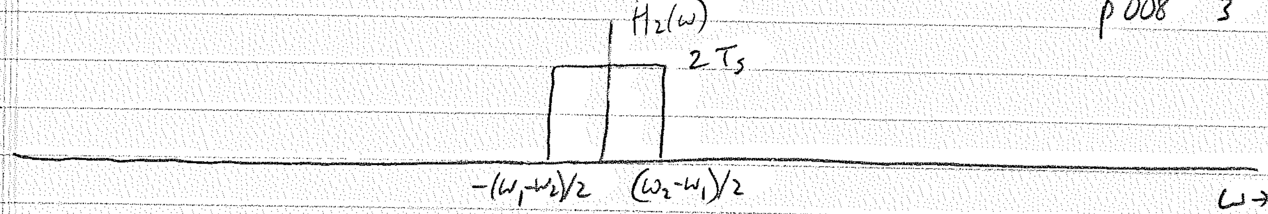


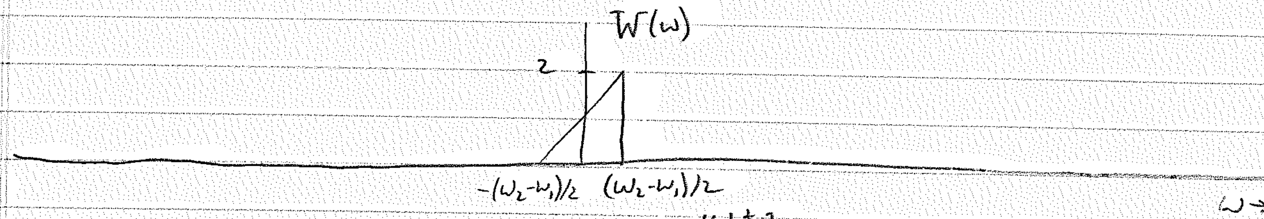


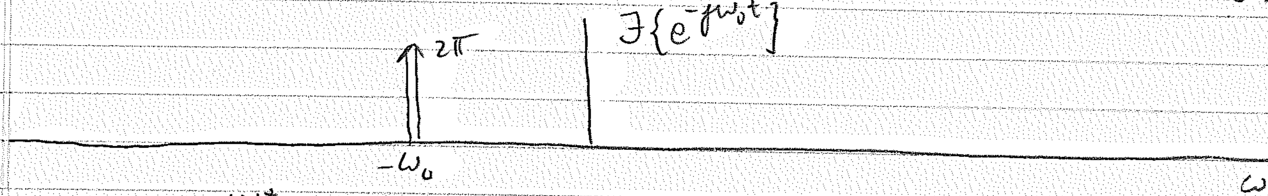
$$x_p(t) = v(t)\, p(t) \leftrightarrow X_p(\omega) = \frac{1}{2\pi} V(\omega) * P(\omega)$$

$$= \frac{1}{2\pi} \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} V(\omega - 2(\omega_2 - \omega_1)n)$$

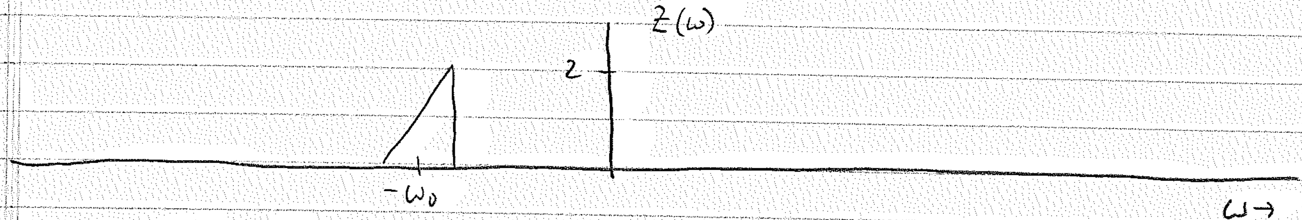






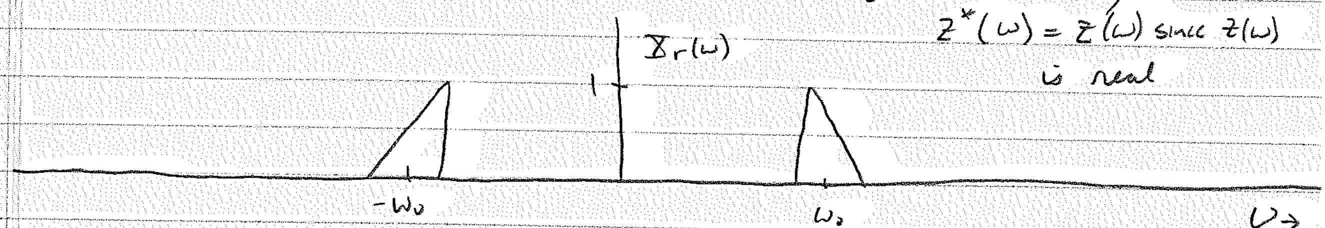


$z(t) = w(t)e^{-j\omega_0 t} \leftrightarrow Z(\omega) = \frac{1}{2\pi} W(\omega) * \mathcal{F}\{e^{-j\omega_0 t}\} = W(\omega+\omega_0)$



$x_r(t) = Re\{z(t)\} \leftrightarrow X_r(\omega) = \frac{1}{2}[Z(\omega) + Z^*(-\omega)]$

in our example, $Z^*(\omega) = Z(\omega)$ since $Z(\omega)$ is real



this is $\frac{1}{2}Z(\omega)$

this is $\frac{1}{2}Z^*(-\omega) = \frac{1}{2}Z(-\omega)$

$Z(\omega)$ is real.

This sampling and reconstruction system works since $x_r(t) = x_c(t)$!

The sampling rate could be reduced to $\omega_s = \omega_2 - \omega_1$.

$$
\begin{tikzcd}
x(t) \arrow[r] & \otimes \arrow[r, "x_r(t)"] & \boxed{H(\omega)} \arrow[r] & y(t) \\
& p(t) \arrow[u] & &
\end{tikzcd}
$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \longleftrightarrow \underline{P}(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T_s} n\right)$$

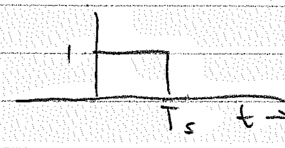
$$x_r(t) = x(t)\,p(t) \longleftrightarrow \underline{X}_r(\omega) = \frac{1}{2\pi} \underline{X}(\omega) * \underline{P}(\omega)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} \underline{X}\left(\omega - \frac{2\pi}{T_s} n\right)$$

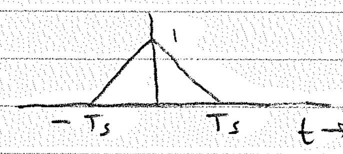
$$x(t) = \exp(-\alpha t)\,u(t) \longleftrightarrow \underline{X}(\omega) = \frac{1}{\alpha + j\omega}$$

$$H_1(\omega) = \begin{cases} T_s & |\omega| \le \frac{2\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

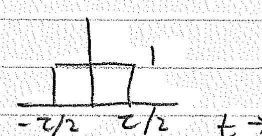
$$h_2(t) = \begin{cases} 1 & 0 \le t \le T_s \\ 0 & \text{otherwise} \end{cases} =$$



$$h_3(t) = \begin{cases} 1 - |t|/T_s & 0 \le |t| < T_s \\ 0 & \text{otherwise} \end{cases} =$$



$$\text{Define} \quad a(t) = \begin{cases} 1 & |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases} =$$



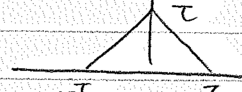
$$\text{Then} \quad A(\omega) = \frac{2\sin(\omega\tau/2)}{\omega} = \frac{\tau}{2}\, \frac{2\sin(\pi\omega\tau/(2\pi))}{\pi\omega\tau/(2\pi)} = \tau\,\mathrm{sinc}(\omega\tau/(2\pi))$$

$$h_2(t) = a(t - T_s/2) \quad \text{where } \tau = T_s$$

$$\updownarrow$$

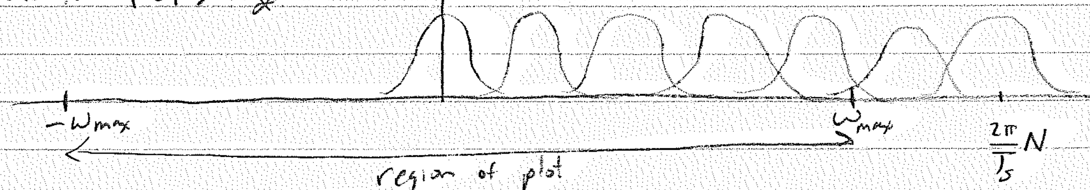
$$H_2(\omega) = \exp(-j\omega T_s/2)\, A(\omega)\Big|_{\tau = T_s} = \exp(-j\omega T_s/2)\, \frac{2\sin(\omega T_s/2)}{\omega}$$

$$a(t) * a(t) =$$



$$h_3(t) = \frac{1}{\tau}\, a(t) * a(t)\Big|_{\tau = T_s} \longleftrightarrow H_3(\omega) = \frac{1}{\tau} A^2(\omega)\Big|_{\tau = T_s} = \frac{1}{T_s}\left[\frac{2\sin(\omega T_s/2)}{\omega}\right]^2$$

Let $X(\omega)$ be essentially zero for $|\omega| > \omega_*$



need to solve

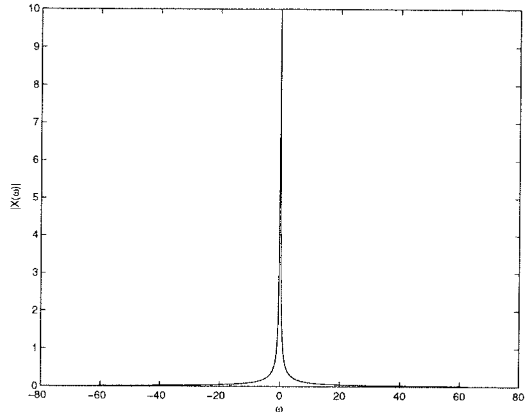
$$\frac{2\pi}{T_s}N - \omega_* \geq \omega_{max}$$

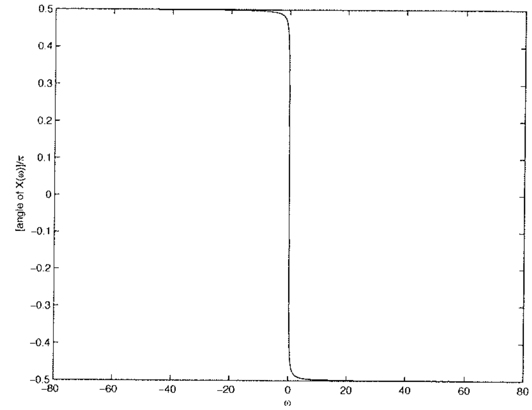
$$\Rightarrow \quad N = \text{ceil}\left(\frac{T_s}{2\pi}\left(\omega_{max} + \omega_* \right) \right)$$

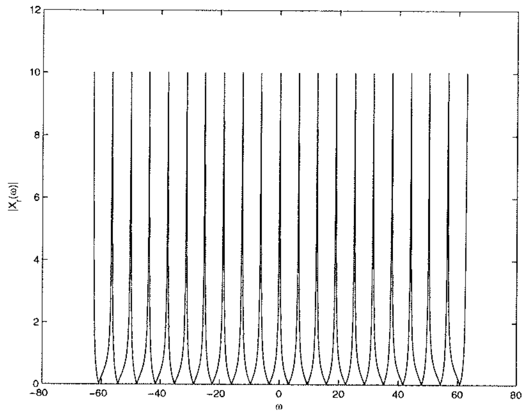
```
function p009
%function p009

Ts=1;
alphascale=0.1;
%alphascale=1.0;
alpha=alphascale/Ts;

omegalimit=10*max(alpha,2*pi/Ts);
omega=[-omegalimit:omegalimit/500:omegalimit];

X=get_X(alpha,omega);
figure;
plot(omega,abs(X),'k')
xlabel('\omega');
ylabel('|X(\omega)|');
print('-deps',['p009.absX.' num2str(alphascale) '.eps']);
figure;
plot(omega,angle(X)./pi,'k')
xlabel('\omega');
ylabel('[angle of X(\omega)]/\pi');
print('-deps',['p009.angleX.' num2str(alphascale) '.eps']);

Xr=get_Xr(alpha,Ts,omega);
figure;
plot(omega,abs(Xr),'k')
xlabel('\omega');
ylabel('|X_r(\omega)|');
print('-deps',['p009.absXr.' num2str(alphascale) '.eps']);
figure;
plot(omega,angle(Xr)./pi,'k')
xlabel('\omega');
ylabel('[angle of X_r(\omega)]/\pi');
print('-deps',['p009.angleXr.' num2str(alphascale) '.eps']);

H1=get_H1(Ts,omega);
figure;
plot(omega,H1,'k')
xlabel('\omega');
ylabel('H_1(\omega)');
print('-deps',['p009.H1.' num2str(alphascale) '.eps']);

Y1=Xr.*H1;
figure;
plot(omega,abs(Y1),'k')
xlabel('\omega');
ylabel('|Y_1(\omega)|');
print('-deps',['p009.absY1.' num2str(alphascale) '.eps']);
figure;
plot(omega,angle(Y1)./pi,'k')
xlabel('\omega');
ylabel('[angle of Y_1(\omega)]/\pi');
print('-deps',['p009.angleY1.' num2str(alphascale) '.eps']);

H2=get_H2(Ts,omega);
figure;
plot(omega,abs(H2),'k')
xlabel('\omega');
ylabel('|H_2(\omega)|');
print('-deps',['p009.absH2.' num2str(alphascale) '.eps']);
figure;
plot(omega,angle(H2)./pi,'k')
xlabel('\omega');
ylabel('[angle of H_2(\omega)]/\pi');
print('-deps',['p009.angleH2.' num2str(alphascale) '.eps']);

Y2=Xr.*H2;
figure;
plot(omega,abs(Y2),'k')
xlabel('\omega');
ylabel('|Y_2(\omega)|');
print('-deps',['p009.absY2.' num2str(alphascale) '.eps']);
figure;
plot(omega,angle(Y2)./pi,'k')
xlabel('\omega');
ylabel('[angle of Y_2(\omega)]/\pi');
print('-deps',['p009.angleY2.' num2str(alphascale) '.eps']);

H3=get_H3(Ts,omega);
figure;
plot(omega,H3,'k')
xlabel('\omega');
ylabel('H_3(\omega)');
print('-deps',['p009.H3.' num2str(alphascale) '.eps']);

Y3=Xr.*H3;
figure;
plot(omega,abs(Y3),'k')
xlabel('\omega');
ylabel('|Y_3(\omega)|');
print('-deps',['p009.absY3.' num2str(alphascale) '.eps']);
figure;
plot(omega,angle(Y3)./pi,'k')
xlabel('\omega');
ylabel('[angle of Y_3(\omega)]/\pi');
print('-deps',['p009.angleY3.' num2str(alphascale) '.eps']);

function Xr=get_Xr(alpha,Ts,omega)
%function Xr=get_Xr(alpha,Ts,omega)
N=ceil((Ts/(2*pi))*(max(abs(omega))+100*alpha)) %approximation: X=0 for |omega|>100*al
pha
summation=zeros(size(omega));
for n=[-N:N]
  summation=summation+get_X(alpha,omega-2*pi*n/Ts);
end
Xr=summation./Ts;

function X=get_X(alpha,omega)
%function X=get_X(alpha,omega)
X=1./(alpha + i.*omega);

function H1=get_H1(Ts,omega)
%function H1=get_H1(Ts,omega)
H1=zeros(size(omega));
indices=find(abs(omega)<=pi/Ts);
H1(indices)=Ts;

function A=get_A(Ts,omega)
%function A=get_A(Ts,omega)
A=Ts.*sinc( omega.*Ts./(2*pi) );

function H2=get_H2(Ts,omega)
%function H2=get_H2(Ts,omega)
H2=exp(-i.*omega.*Ts./2).*get_A(Ts,omega);
```

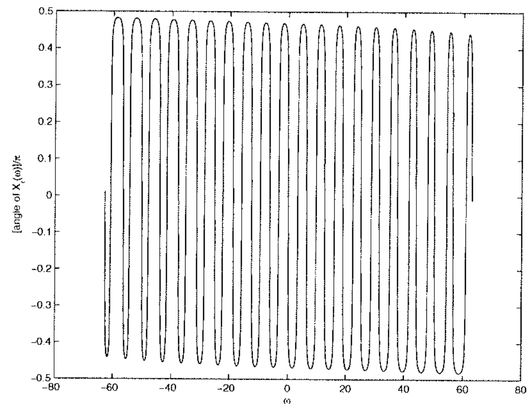
```
function H3=get_H3(Ts,omega)
%function H3=get_H3(Ts,omega)
H3=(get_A(Ts,omega).^2)./Ts;
```

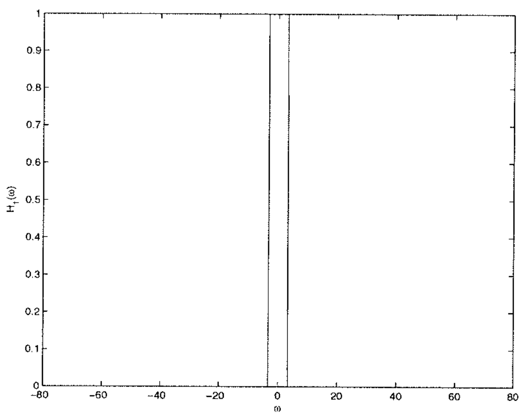
$\alpha = (1/10)(1/T_s)$:

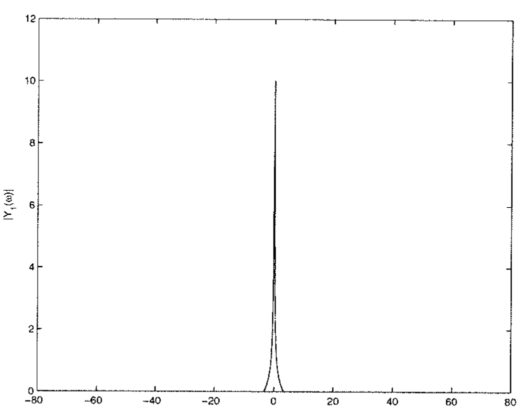


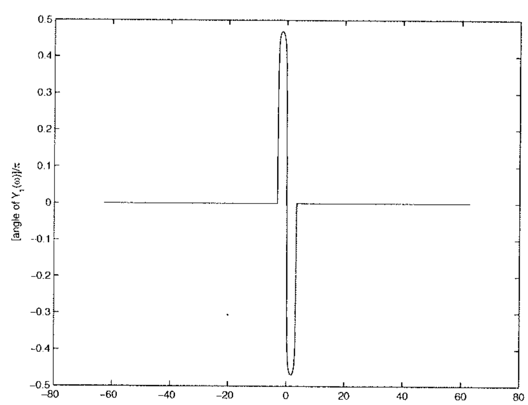












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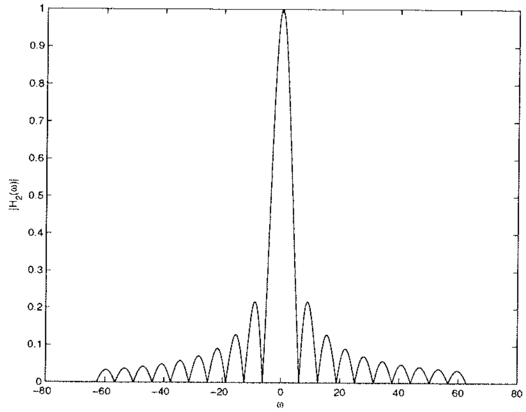

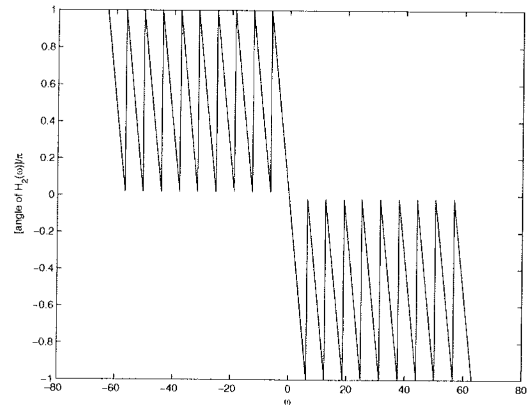

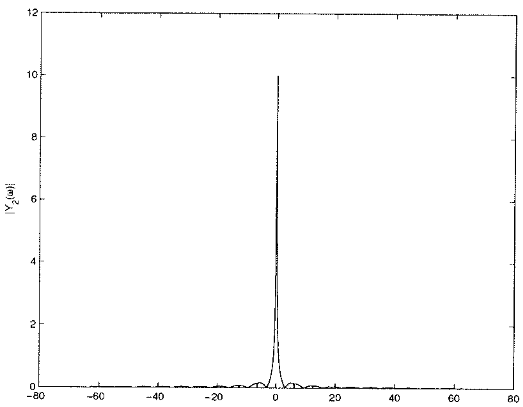

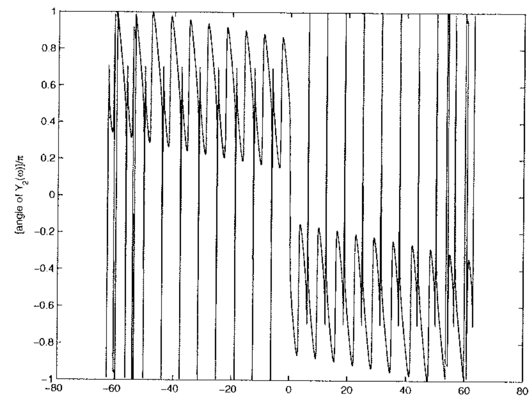

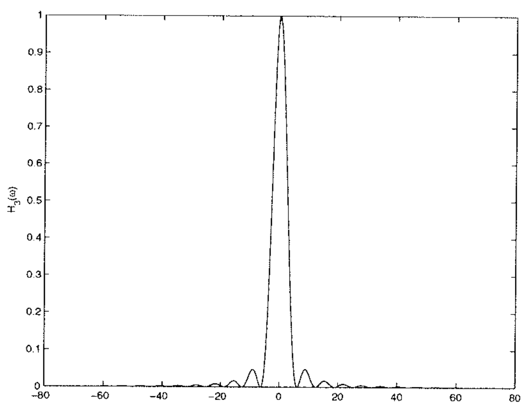

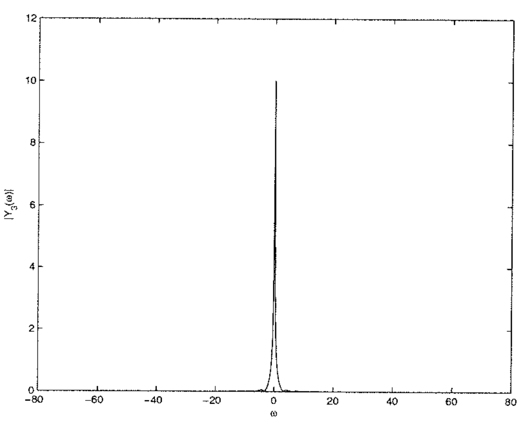

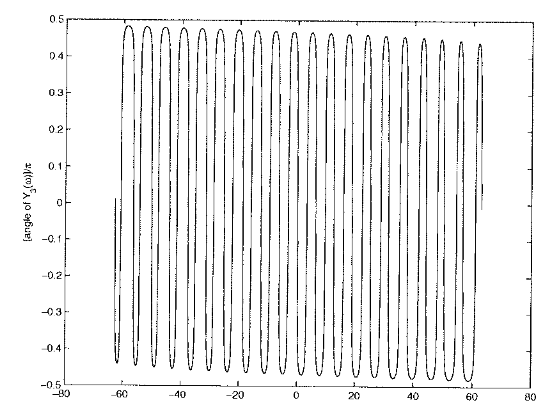

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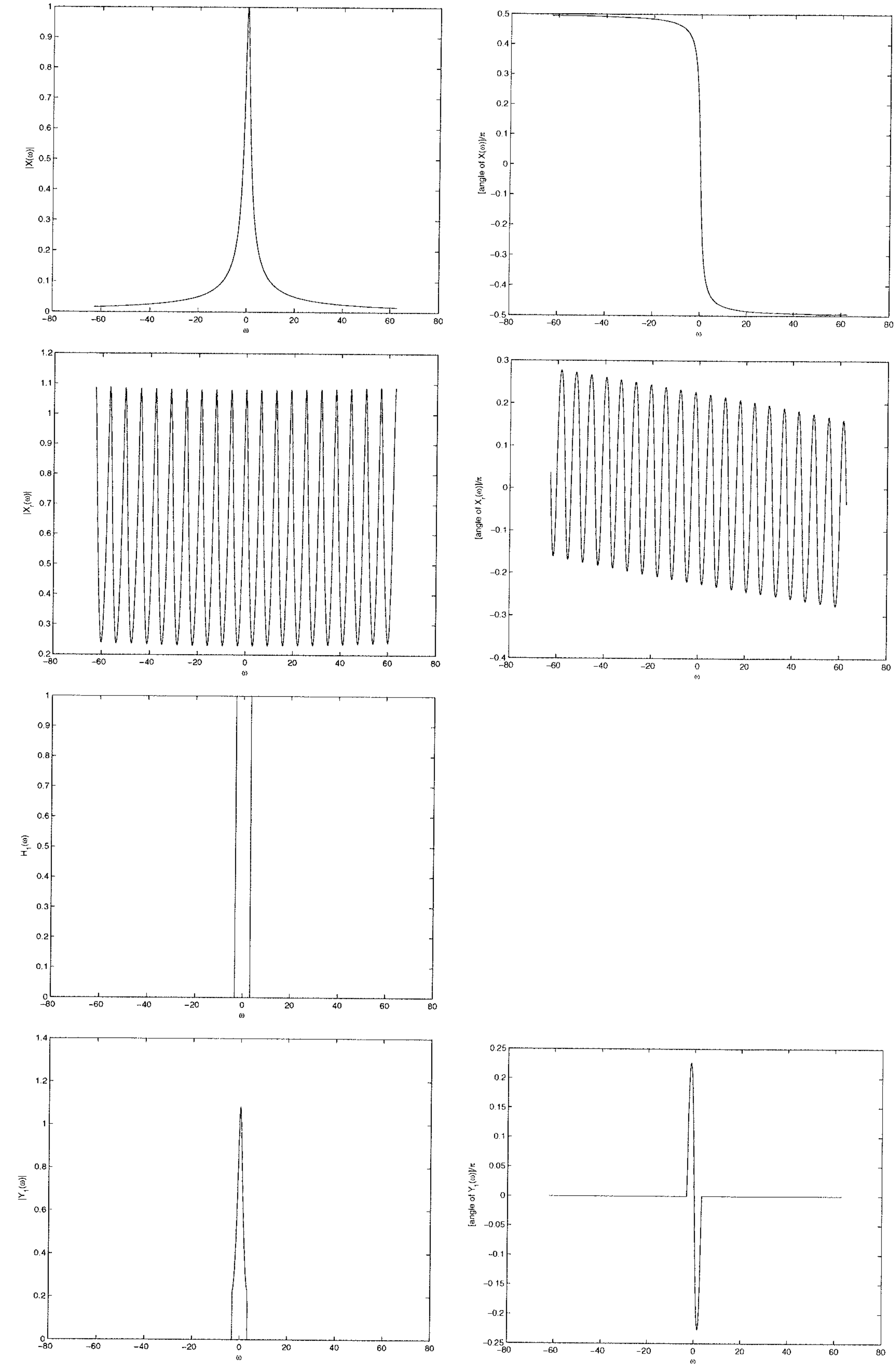
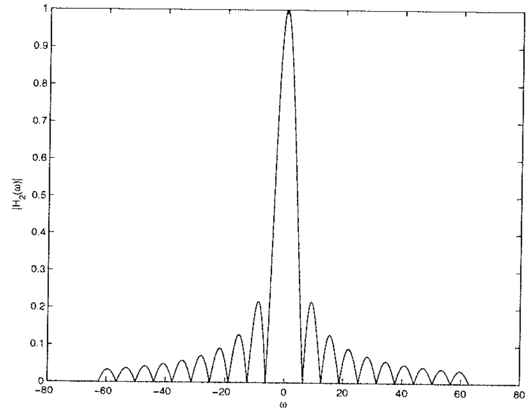
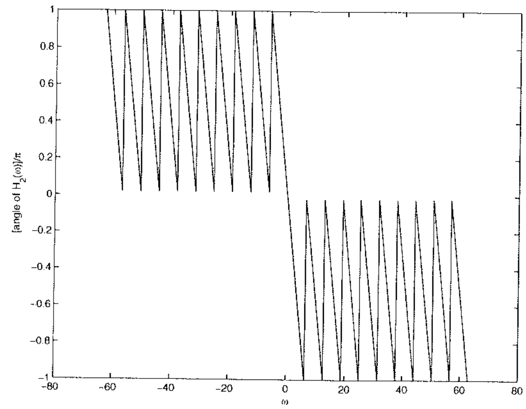
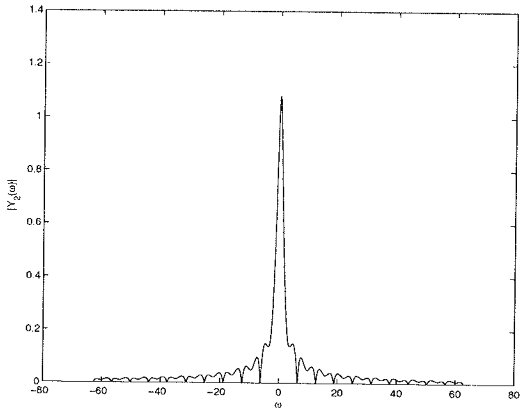
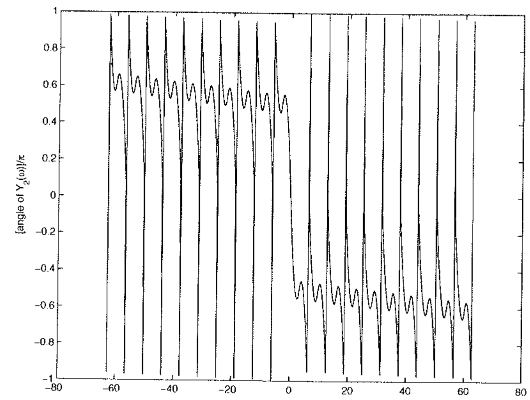


$\alpha = 1/T_s$:


|H_2(ω)|
ω


[angle of H_2(ω)]/π
ω


|Y_2(ω)|
ω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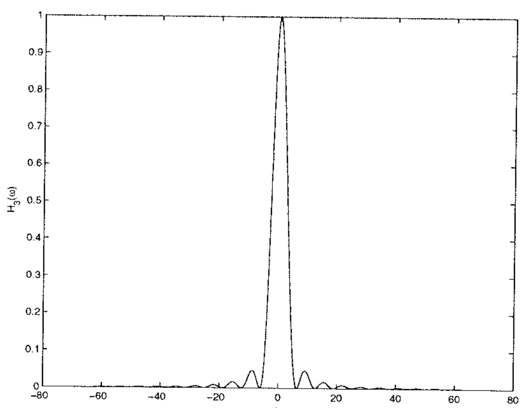
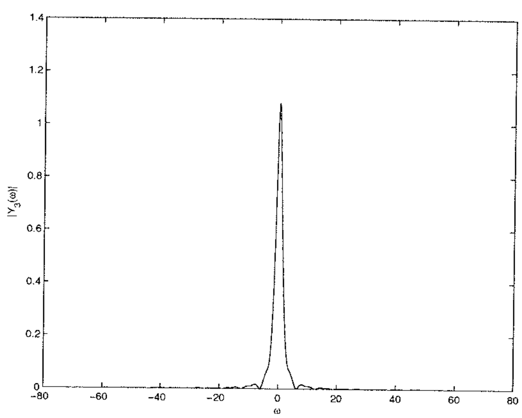
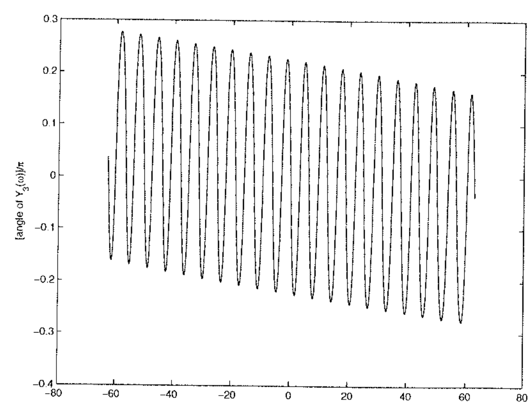
ω


H_3(ω)
ω


|Y_3(ω)|
ω


[angle of Y_3(ω)]/π
ω