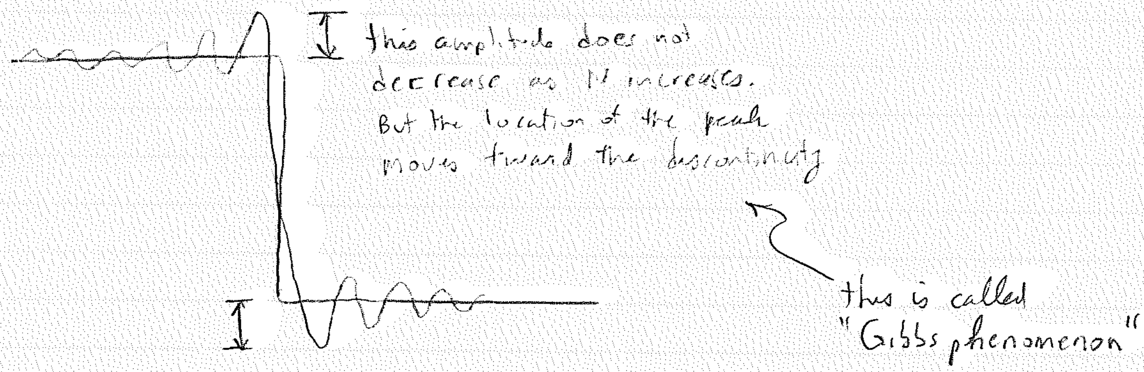
a) The amplitude of the oscillations at the discontinuity of $x(t)$ do not decrease:




b) It appears that the value of the reconstruction at the discontinuity is $1/2$, which is the average of the value of $x(t)$ just before and just after the discontinuity.

$$x(t) = m(t)\sqrt{2}\cos 2\pi f_c t$$

$$x_d(t) = x(t-\tau) = m(t-\tau)\sqrt{2}\cos 2\pi f_c(t-\tau)$$

$$v(t) = x_d(t)\sqrt{2}\cos 2\pi f_c t = 2m(t-\tau)\cos 2\pi f_c(t-\tau)\cos 2\pi f_c t$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}\cos(\alpha-\beta) + \tfrac{1}{2}\cos(\alpha+\beta)$$

$$\Rightarrow v(t) = m(t-\tau)\Big(\underbrace{\cos 2\pi f_c \tau}_{\text{this is a constant independent of } t.} + \cos 2\pi\, 2f_c(t - \tau/2)\Big)$$

Compute $V(\omega)$:

$$m(t-\tau) \longleftrightarrow \exp(-j\omega\tau)\, M(\omega)$$

$$\cos 2\pi\, 2f_c(t-\tau/2) \longleftrightarrow \exp(-j\omega\tau/2)\,\tfrac{1}{2}\Big[\delta(\omega - 2\pi\, 2f_c) + \delta(\omega + 2\pi\, 2f_c)\Big]$$

$$= \tfrac{1}{2}\exp(-j\, 2\pi\, 2f_c\, \tau/2)\,\delta(\omega - 2\pi\, 2f_c) + \tfrac{1}{2}\exp\left(j\, 2\pi\, 2f_c \tfrac{\tau}{2}\right)\delta(\omega + 2\pi 2$$

$$= \underbrace{\tfrac{1}{2}\exp(-j 2\pi f_c \tau)\,\delta(\omega - 2\pi 2f_c) + \tfrac{1}{2}\exp(j\, 2\pi f_c \tau)}_{\text{constants independent of } \omega}\delta(\omega + 2\pi 2$$

therefore,

$$V(\omega) = \exp(-j\omega\tau)\, M(\omega)\cos 2\pi f_c \tau + \frac{1}{2\pi}\exp(-j\omega\tau)M(\omega) * \left[\begin{array}{l} \tfrac{1}{2}\exp(-j 2\pi f_c \tau)\,\delta(\omega - 2\pi 2f_c) \\ \quad + \tfrac{1}{2}\exp(j 2\pi f_c \tau)\,\delta(\omega + 2\pi\, 2f_c) \end{array}\right]$$

$$= \exp(-j\omega\tau)\, M(\omega)\cos 2\pi f_c \tau$$

$$+ \frac{1}{4\pi}\exp(-j(\omega - 2\pi 2f_c)\tau)\, M(\omega - 2\pi 2f_c)\exp(-j 2\pi f_c \tau)$$

$$+ \frac{1}{4\pi}\exp(-j(\omega + 2\pi 2f_c)\tau)\, M(\omega + 2\pi 2f_c)\exp(j 2\pi f_c \tau)$$

$$= \exp(-j\omega\tau)\, M(\omega)\cos 2\pi f_c\tau \qquad (1)$$

$$+ \frac{1}{4\pi}\exp(-j(\omega - 2\pi f_c)\tau)\, M(\omega - 2\pi 2f_c) \qquad (2)$$

$$+ \frac{1}{4\pi}\exp(-j(\omega + 2\pi f_c)\tau)\, M(\omega + 2\pi 2f_c) \qquad (3)$$

$M(\omega) = 0$ for all $|\omega| > W$ (A)

$H(\omega) = 0$ for all $|\omega| > W$ and $H(\omega) = 1$ for all $|\omega| \leq W$ (B)

(A) $\Rightarrow$ Term (1) is non zero only for $|\omega| < W$. So it passes through the filter unaltered.

(A) $\Rightarrow$ $\left\{ \begin{array}{l} \text{Term (1)} \\ \text{Term (2)} \end{array} \right\}$ is non zero only for $\left\{ \begin{array}{l} |\omega - 2\pi 2f_c| < W \\ |\omega + 2\pi 2f_c| < W \end{array} \right\}$. So it is completely removed by the filter.

Therefore

$$Y(\omega) = \exp(-j\omega\tau) M(\omega) \underbrace{\cos 2\pi f_c \tau}_{\text{constant independent of } \omega}$$

$$\Updownarrow$$

$$y(t) = m(t - \tau) \underbrace{\cos 2\pi f_c \tau}_{\text{constant independent of } t}$$

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$p(t) = \sqrt{2}\cos 2\pi f_c t$

$u(t) = x(t) + p(t)$

$$y(t) = [u(t)]^2 = [x(t)+p(t)]^2 = x^2(t) + 2x(t)p(t) + p^2(t)$$

$$= x^2(t) + 2x(t)p(t) + [\sqrt{2}\cos 2\pi f_c t]^2 = x^2(t) + 2x(t)p(t) + 2\tfrac{1}{2}(1+\cos 2\pi 2f_c t)$$

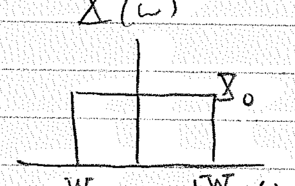
$$= x^2(t) + 2x(t)\sqrt{2}\cos 2\pi f_c t + 1 + \cos 2\pi 2 f_c t$$

$$Y(\omega) = \frac{1}{2\pi}X(\omega) * X(\omega) + \frac{1}{2\pi}2X(\omega) * \sqrt{2}\pi[\delta(\omega-2\pi f_c) + \delta(\omega+2\pi f_c)]$$

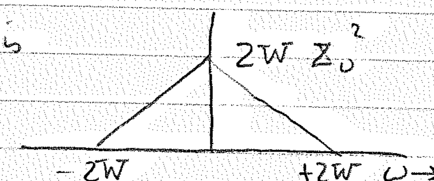
$$+ \delta(\omega) + \pi[\delta(\omega - 2\pi 2f_c) + \delta(\omega + 2\pi 2 f_c)]$$

$$= \frac{1}{2\pi}X(\omega) * X(\omega) + \sqrt{2}[X(\omega - 2\pi f_c) + X(\omega + 2\pi f_c)]$$

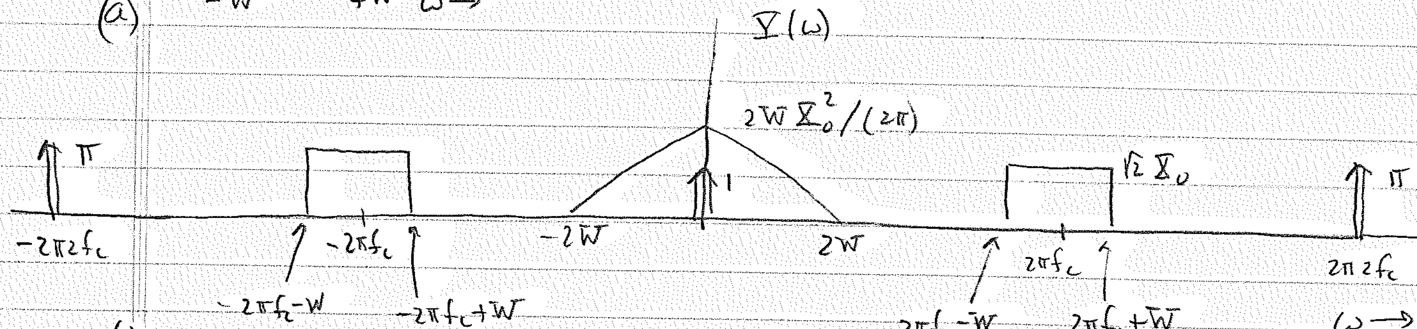
$$+ \delta(\omega) + \pi[\delta(\omega - 2\pi 2 f_c) + \delta(\omega + 2\pi 2 f_c)]$$



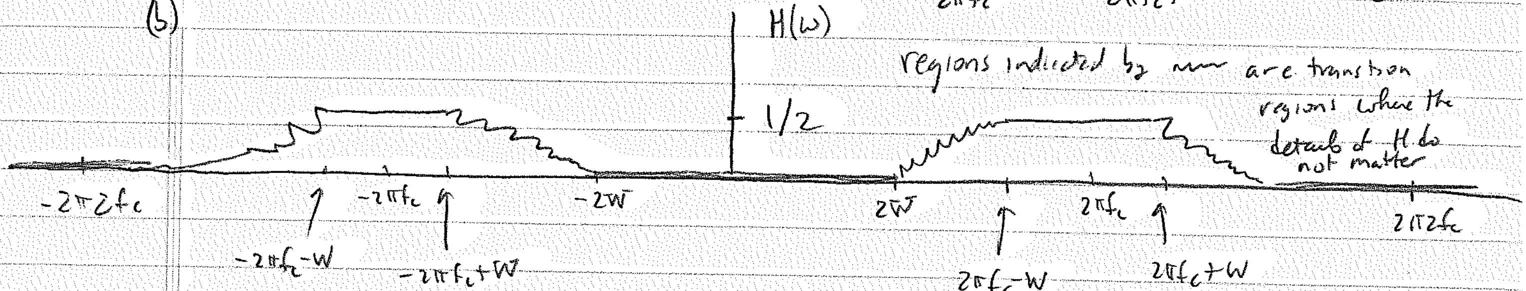

therefore $X(\omega) * X(\omega)$ is




(a)




(b)




(c) This filter will work for any $x(t)$ in the class of signals that are bandlimited to $\pm W$

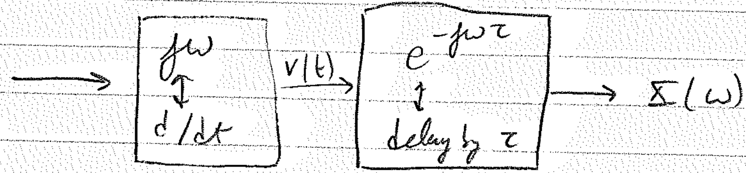
(a) $X(\omega) = \dfrac{j\omega}{0.1 + j\omega} e^{-j\omega 0.2}$

$= \dfrac{j\omega}{\alpha + j\omega} e^{-j\omega\tau}$

$M(\omega) = \dfrac{1}{\alpha + j\omega}$

$\updownarrow$

$m(t) = e^{-\alpha t} u(t)$




$v(t) = \dfrac{d}{dt}\left[e^{-\alpha t} u(t)\right] = \dfrac{d e^{-\alpha t}}{dt} u(t) + e^{-\alpha t} \dfrac{du}{dt}$

$= -\alpha e^{-\alpha t} u(t) + e^{-\alpha t} \delta(t)$

$= -\alpha e^{-\alpha t} u(t) + \delta(t) \quad \left(\text{since } e^{-\alpha t}\Big|_{t=0} = 1\right)$

$x(t) = v(t-\tau)$

$= -\alpha e^{-\alpha(t-\tau)} u(t-\tau) + \delta(t-\tau)$

(b) $X(\omega) = 2 + 2\cos(\omega) = 2 + 2\,\frac{1}{2}\left(e^{j\omega} + e^{-j\omega}\right)$

$= 2 + e^{j\omega} + e^{-j\omega}$

$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

$x(t) = 2\delta(t) + \delta(t+1) + \delta(t-1)$

(c) $X(\omega) = \dfrac{1}{1+j\omega} - \dfrac{1}{2+j\omega}$

$\updownarrow \quad \updownarrow \quad \updownarrow$

$x(t) = e^{-t} u(t) - e^{-2t} u(t) = \left(e^{-t} - e^{-2t}\right) u(t)$

$h(t)$ real

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt$$

$$H^*(\omega) = \left[\int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt\right]^* = \int_{-\infty}^{+\infty} \left[h(t)\, e^{-j\omega t}\right]^* dt$$

$$= \int_{-\infty}^{+\infty} h^*(t) \left(e^{-j\omega t}\right)^* dt = \int_{-\infty}^{+\infty} h^*(t)\, e^{+j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} h^*(t)\, e^{-j(-\omega)t}\, dt \qquad (*)$$

$h(t) \in \mathbb{R}$

$$= \int_{-\infty}^{+\infty} h(t)\, e^{-j(-\omega)t}\, dt$$

$$= H(-\omega)$$

$|H(\omega)| = |H^*(\omega)| = |H(-\omega)|$ i.e. $|H(\omega)|$ is even

$\measuredangle H(\omega) = -\measuredangle H^*(\omega) = -\measuredangle H(-\omega)$ i.e. $\measuredangle H(\omega)$ is odd

$h(t)$ pure imaginary

$$H^*(\omega) = \cdots = \int_{-\infty}^{+\infty} h^*(t)\, e^{-j(-\omega)t}\, dt \quad \text{by } (*)$$

$h(t)$ pure imaginary

$$= -\int_{-\infty}^{+\infty} h(t)\, e^{-j(-\omega)t}\, dt$$

$$= -H(-\omega).$$

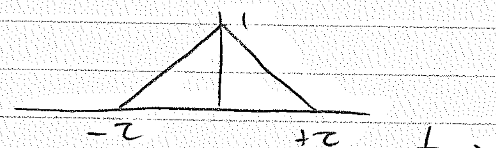
$|H(\omega)| = |H^*(\omega)| = |-H(-\omega)| = |H(-\omega)|$ i.e. $|H(\omega)|$ is even

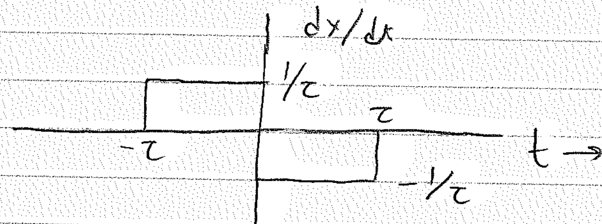
$$\measuredangle H(\omega) = -\measuredangle H^*(\omega) = -\measuredangle -H(-\omega) = -\measuredangle\left[(-1)(H(-\omega))\right]$$

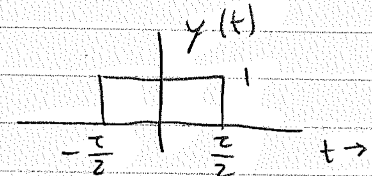
$$= -\left[\measuredangle(-1) + \measuredangle H(-\omega)\right] = -\left[\pi + \measuredangle H(-\omega)\right]$$

$$= -\measuredangle H(-\omega) - \pi$$

$$x(t) = \begin{cases} 1 - |t|/\tau & |t|<\tau \\ 0 & \text{otherwise} \end{cases}$$

$=$ 

(a) 

Define $y(t)$ by . A standard result is that

$$Y(\omega) = \frac{2\sin(\omega\tau/2)}{\omega} = \frac{\pi\tau}{2\pi}\,\frac{2\sin(\pi\,\omega\tau/(2\pi))}{\omega\,\pi\tau/(2\pi)}$$

$$= \tau\,\frac{\sin(\pi\,\omega\tau/(2\pi))}{\pi\,\omega\tau/(2\pi)} = \tau\,\mathrm{sinc}(\omega\tau/(2\pi))$$

$$\frac{dx}{dt} = \frac{1}{\tau}\,y\left(t+\frac{\tau}{2}\right) - \frac{1}{\tau}\,y\left(t-\frac{\tau}{2}\right)$$

$\updownarrow$

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = \frac{1}{\tau}e^{j\omega\frac{\tau}{2}}\,Y(\omega) - \frac{1}{\tau}e^{-j\omega\frac{\tau}{2}}\,Y(\omega) = \frac{1}{\tau}\left\{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right\}Y(\omega)$$

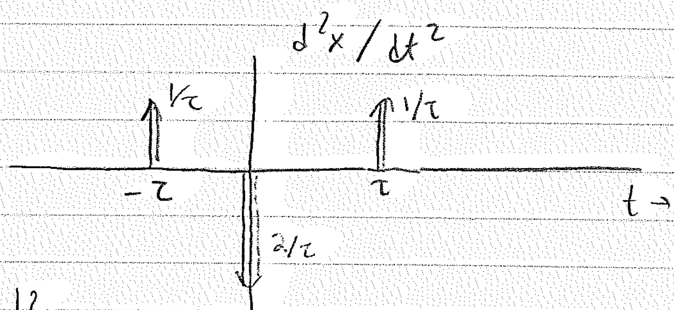
$$= \frac{2j}{\tau}\sin\omega\tau/2\;Y(\omega) = \frac{2j}{\tau}\sin(\omega\tau/2)\,\frac{2\sin(\omega\tau/2)}{\omega}$$

$$= \frac{j\omega}{\tau}\left[\frac{2\sin(\omega\tau/2)}{\omega}\right]^2$$

but also, $\mathcal{F}\left\{\frac{dx}{dt}\right\} = j\omega\,\mathcal{F}\{x(t)\} \Longrightarrow X(\omega) = \frac{1}{j\omega}\,\mathcal{F}\left\{\frac{dx}{dt}\right\}$.

So $X(\omega) = \frac{1}{j\omega}\,\frac{j\omega}{\tau}\left[\frac{2\sin(\omega\tau/2)}{\omega}\right]^2 = \frac{1}{\tau}\left[\frac{2\sin(\omega\tau/2)}{\omega}\right]^2$

(b)




$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$
$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ (*)

$$\frac{d^2x}{dt^2} = \frac{1}{\tau}\left[\delta(t+\tau) - 2\delta(t) + \delta(t-\tau)\right]$$

$$\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} = \frac{1}{\tau}\left[e^{j\omega\tau} - 2 + e^{-j\omega\tau}\right] = \frac{1}{\tau}\left[2\cos\omega\tau - 2\right] = \frac{2}{\tau}(\cos\omega\tau - 1)$$

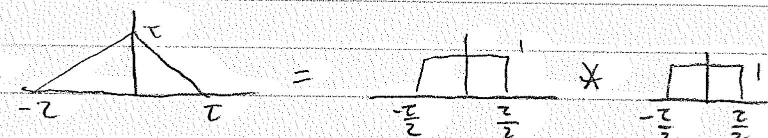
by (*) $\quad = -\frac{2}{\tau}2\sin^2(\omega\tau/2) = -\frac{4}{\tau}\sin^2(\omega\tau/2)$

but also, $\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} = (j\omega)^2\,\mathcal{F}\{x(t)\}$

$$\Longrightarrow X(\omega) = \frac{1}{(j\omega)^2}\,\frac{-4}{\tau}\sin^2(\omega\tau/2) = \frac{1}{\tau}\left[\frac{2\sin(\omega\tau/2)}{\omega}\right]^2$$

This is the same answer as was determined in Part (a).

Third method:

$\tau\,x(t) =$  $= y(t) * y(t)$


therefore

$$X(\omega) = \frac{1}{\tau}\left[Y(\omega)\right]^2 = \frac{1}{\tau}\left[\frac{2\sin(\omega\tau/2)}{\omega}\right]^2$$

Fourth method:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt = \int_{-\tau}^{0}(1 + t/\tau)e^{-j\omega t}\,dt + \int_{0}^{\tau}(1 - t/\tau)e^{-j\omega t}\,dt$$

= integration by parts = (hopefully) the same answer.