ECE 2200 / Spring 2016 / HW #1 Solutions

P. 2-6 (SP-1) = P. 2-6 (DSP-2)

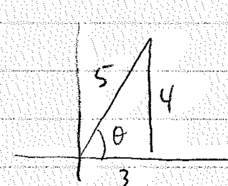
$$(\cos\theta + j\sin\theta)^n = \left(\frac{e^{j\theta} + e^{-j\theta}}{2} + j\,\frac{e^{j\theta} + e^{-j\theta}}{2j} \right)^n = \left(e^{j\theta}\right)^n$$

$$= e^{jn\theta} = \cos(n\theta) + j\sin(n\theta)$$

$$\left(\frac{3}{5} + j\frac{4}{5}\right)^{100}:$$

$$\left|\frac{3}{5} + j\frac{4}{5}\right|^2 = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9+16}{25} = 1$$

$$\measuredangle\left(\frac{3}{5} + j\frac{4}{5}\right) = \arctan\left(\frac{4/5}{3/5}\right) = \arctan\left(\frac{4}{3}\right) = 53.13^\circ$$



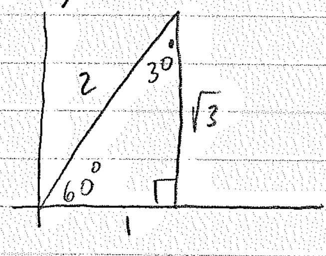
$\tan\theta = \frac{4}{3}$ $\quad \theta = 53.13^\circ$

$\theta = .927$ rad

$$\left(\frac{3}{5} + j\frac{4}{5}\right)^{100} = \left(e^{j.927}\right)^{100} = e^{j92.7} = .0230 - j.999$$

P. 2-7 (SP-1) = P. 2-7 (DSP-2) $\frac{\pi}{3}$ rad $= 60°$, $\frac{\pi}{6}$ rad $= 30°$

a) $3e^{j\frac{\pi}{3}} + 4e^{-j\frac{\pi}{6}}$



$\cos 60° = \frac{1}{2}$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\sin 30° = \frac{1}{2}$

$= 3\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) + 4\left(\cos\frac{-\pi}{6} + j\sin\frac{-\pi}{6}\right)$

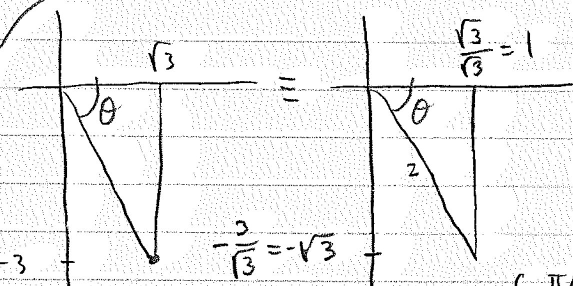
$= 3\left(\cos 60° + j\sin 60°\right) + 4\left(\cos 30° - j\sin 30°\right)$

$= 3\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + 4\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)$

$= \frac{3}{2} + j\frac{3\sqrt{3}}{2} + 2\sqrt{3} - j2$

$= \left(\frac{3}{2} + 2\sqrt{3}\right) + j\left(\frac{3\sqrt{3}}{2} - 2\right)$

$= 4.96 + j.598$



b) $(\sqrt{3} - j3)^{10}$

$\theta = \begin{cases} -\pi/3 \text{ rad} \\ -60° \end{cases}$

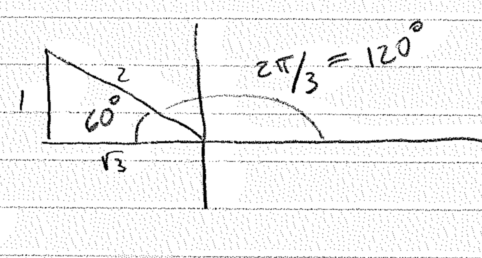
$|\sqrt{3} - j3| = \sqrt{(\sqrt{3})^2 + (-3)^2} = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

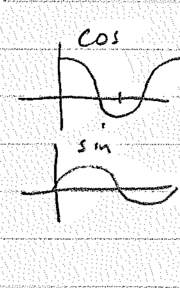
$\measuredangle(\sqrt{3} - j3) = -\arctan\left(\frac{3}{\sqrt{3}}\right) = -\arctan(\sqrt{3}) = \begin{cases} -\pi/3 \text{ rad} \\ -60° \end{cases}$

$(\sqrt{3} - j3)^{10} = \left(2\sqrt{3}e^{-j\frac{\pi}{3}}\right)^{10} = 2^{10}(\sqrt{3})^{10} e^{-j\frac{10\pi}{3}}$

$= 2^{10}3^5 \exp\left(-j\left(3 + \frac{1}{3}\right)\pi\right) = 2^{10}3^5 \exp\left(-j\left[\left(3+\frac{1}{3}\right)\pi - 4\pi\right]\right)$

$= 2^{10}3^5 \exp\left(+j\frac{2\pi}{3}\right)$





$= 2^{10}3^5\left(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right)$

$= 2^{10}3^5\left(-\cos\frac{1}{3}\pi + j\sin\frac{1}{3}\pi\right)$

$= 2^{10}3^5\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 2^9 3^5\left(-1 + j\sqrt{3}\right) = 124416\left(-1 + j\sqrt{3}\right)$

c) $(\sqrt{3} - j3)^{-1} = \left(2\sqrt{3}\, e^{-j\pi/3}\right)^{-1}$

Part (b)

$= \frac{1}{2\sqrt{3}} e^{+j\pi/3} = \frac{1}{2\sqrt{3}}\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = \frac{1}{2\sqrt{3}}\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \frac{1}{4\sqrt{3}}\left(1 + j\sqrt{3}\right)$

$= .144 + j.25$

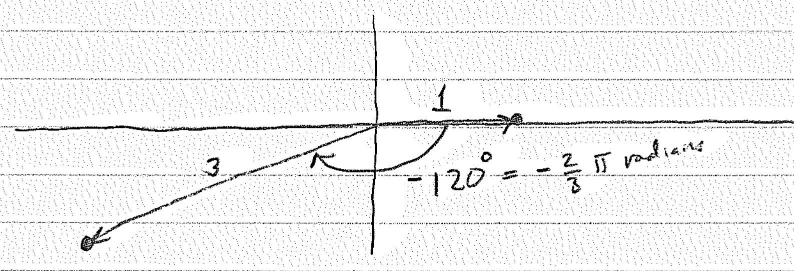
P-2.10 (DSP-2)

$$x(t) = 3\cos(\omega_0 t - \tfrac{2}{3}\pi) + \cos(\omega_0 t)$$

$$= 3\,\mathrm{Re}\left\{ e^{j(\omega_0 t - \frac{2}{3}\pi)} \right\} + \mathrm{Re}\left\{ e^{j\omega_0 t} \right\}$$

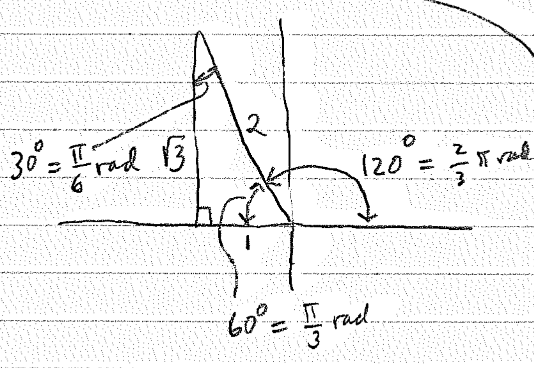
$$= \mathrm{Re}\left\{ 3e^{j(\omega_0 t - \frac{2}{3}\pi)} + e^{j\omega_0 t} \right\}$$

$$= \mathrm{Re}\left\{ e^{j\omega_0 t} \left[3e^{-j\frac{2}{3}\pi} + 1 \right] \right\}$$



$$3e^{-j\frac{2}{3}\pi} + 1 = 3\left[\cos(-\tfrac{2}{3}\pi) + j\sin(-\tfrac{2}{3}\pi)\right] + 1$$

$$= 3\cos(\tfrac{2}{3}\pi) + 1 - j\,3\sin(\tfrac{2}{3}\pi)$$



$$= 3\left(\tfrac{-1}{2}\right) + 1 - j\,3\,\tfrac{\sqrt{3}}{2}$$

$$= -\tfrac{1}{2} - j\,3\,\tfrac{\sqrt{3}}{2}$$

$$= 2.646\,e^{-j1.761\,\text{rad}}$$

$$\cos\tfrac{2}{3}\pi = -\tfrac{1}{2} \qquad \sin\tfrac{2}{3}\pi = \tfrac{\sqrt{3}}{2}$$

So

(a) $$x(t) = \mathrm{Re}\left\{ e^{j\omega_0 t} \left[3e^{-j\frac{2}{3}\pi} + 1 \right] \right\}$$

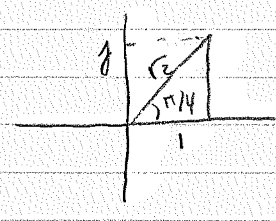
$$= \mathrm{Re}\left\{ e^{j\omega_0 t}\, 2.646\, e^{-j1.761} \right\}$$

$$= 2.646\cos(\omega_0 t - 1.761)$$

(b) $z(t) = 2.646\,e^{j(\omega_0 t - 1.761)}$, $x(t) = \mathrm{Re}\{z(t)\}$

P_2.12 (DSP-2)

$$\mathrm{Re}\{(1+j)e^{j\theta}\} = 0$$

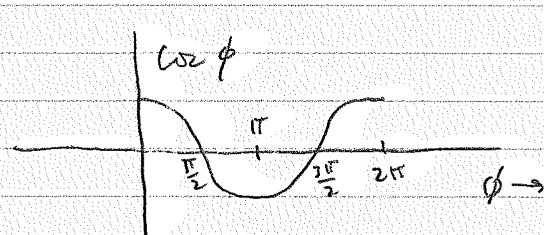


$$\Leftrightarrow \quad \mathrm{Re}\{\sqrt{2}\, e^{j\pi/4}\, e^{j\theta}\} = 0$$

$$\Leftrightarrow \quad \mathrm{Re}\{\sqrt{2}\, e^{j(\theta + \pi/4)}\} = 0$$

$$\Leftrightarrow \quad \sqrt{2}\cos(\theta + \pi/4) = 0$$

$$\Leftrightarrow \quad \cos(\theta + \pi/4) = 0$$

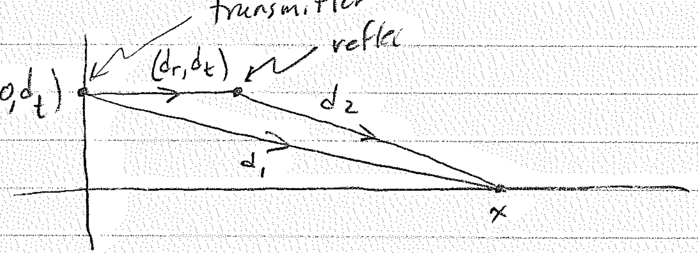


$$\cos\left(\frac{\pi}{2} + n\pi\right) = 0 \quad \text{for } n \in \mathbb{Z}$$

$$\Leftrightarrow \quad \theta + \pi/4 = \pi/2 + n\pi \quad \text{for } n \in \mathbb{Z}$$

$$\Leftrightarrow \quad \theta = \pi/4 + n\pi \quad \text{for } n \in \mathbb{Z}$$

P-2.21 (SP-1) = P-2.28 (DSP-2)   1



$d_t = 10^3$ m $d_r = 55$ m

c = speed of light = 3×10^8 m/s

$f_c = 150\times10^6$ Hz $A = 1$

$x \longrightarrow$

$$d_1 = \sqrt{x^2 + d_t^2} \qquad d_2 = d_r + \sqrt{(x-d_r)^2 + d_t^2}$$

(a) $t_1(x) = d_1(x)/c = \sqrt{x^2+d_t^2}\,/c \qquad t_2(x) = d_2(x)/c = \left[d_r + \sqrt{(x-d_r)^2+d_t^2}\right]/c$

(b) + (c) $s(t) = A\cos(2\pi f_c t)$

$$r(t) = s(t-t_1) + s(t-t_2)$$

$$= A\cos(2\pi f_c(t-t_1)) + A\cos(2\pi f_c(t-t_2))$$

$$= A\,\mathrm{Re}\left\{e^{j2\pi f_c(t-t_1)}\right\} + A\,\mathrm{Re}\left\{e^{j2\pi f_c(t-t_2)}\right\}$$

$$= A\,\mathrm{Re}\left\{e^{j2\pi f_c(t-t_1)} + e^{j2\pi f_c(t-t_2)}\right\}$$

$$= A\,\mathrm{Re}\left\{e^{j2\pi f_c t}\left[e^{-j2\pi f_c t_1} + e^{-j2\pi f_c t_2}\right]\right\}$$

$$= A\,\mathrm{Re}\left\{e^{j2\pi f_c t}\, e^{-j2\pi f_c \frac{t_1+t_2}{2}}\left[e^{-j2\pi f_c \frac{t_1-t_2}{2}} + e^{+j2\pi f_c \frac{t_1-t_2}{2}}\right]\right\}$$

$$= A\,\mathrm{Re}\left\{\exp\left(j2\pi f_c\left[t - \frac{t_1+t_2}{2}\right]\right) 2\cos\left(2\pi f_c \frac{t_1-t_2}{2}\right)\right\}$$

$$= 2A\cos\left(2\pi f_c \frac{t_1-t_2}{2}\right)\mathrm{Re}\left\{\exp\left(j2\pi f_c\left[t - \frac{t_1+t_2}{2}\right]\right)\right\}$$

$$= \underbrace{2A\cos\left(2\pi f_c \frac{t_1-t_2}{2}\right)}_{\text{Amplitude - no dependence on } t}\cos\left(2\pi f_c\left(t - \frac{t_1+t_2}{2}\right)\right)$$

↑ dependence on t

(b) Set $x = 0$. Then $t_1 = d_t/c \qquad t_2 = \left[d_r + \sqrt{d_r^2 + d_t^2}\right]/c$

$= 3.33\times10^{-6}$ s $= 3.52\times10^{-6}$ s

$\frac{t_1+t_2}{2} = 3.42\times10^{-6}$ s $\frac{t_1-t_2}{2} = -9.41\times10^{-8}$ s

amplitude $= (2)(1)\cos\left(2\pi f_c \frac{t_1-t_2}{2}\right) = 1.389$ frequency $= f_c = 150\times10^6$ Hz

phase $= -2\pi f_c \frac{t_1+t_2}{2} = -2\pi\ 150\times10^6 \times 3.42\times10^{-6} = 3230$ rad $= .803$ rad

↑ subtract $(514)(2\pi)$

P-2.21 (SP-1) = P-2.28 (DSP-2)

c) find a position x such that $r(t)$ is zero for all t.

"all t" $\Rightarrow$ need amplitude to equal zero

$$\Rightarrow \cos\left(2\pi f_c \frac{t_1 - t_2}{2}\right) = 0$$

$$\Rightarrow 2\pi f_c \frac{t_1 - t_2}{2} = \frac{\pi}{2} + m\pi \text{ for some } m \in \mathbb{Z}$$

$$\Rightarrow 2\pi f_c \frac{d_1 - d_2}{2c} = \frac{\pi}{2} + m\pi$$

Since $d_2 > d_1$, we will get real valued roots for x only for $m = -1, -2, -3, \ldots$

$$\Rightarrow d_1 - d_2 = \frac{c}{f_c}\left(\frac{1}{2} + m\right)$$

$$\Rightarrow d_1 - \frac{c}{f_c}\left(\frac{1}{2} + m\right) = d_2$$

$$\Rightarrow \sqrt{x^2 + d_t^2} - \frac{c}{f_c}\left(\frac{1}{2} + m\right) = d_r + \sqrt{(x - d_r)^2 + d_t^2}$$

$$\Rightarrow \sqrt{x^2 + d_t^2} - \underbrace{\left[\frac{c}{f_c}\left(\frac{1}{2} + m\right) + d_r\right]}_{:= \xi} = \sqrt{(x - d_r)^2 + d_t^2}$$

Square both sides

$$\left[\sqrt{x^2 + d_t^2} - \xi\right]^2 = (x - d_r)^2 + d_t^2$$

$$x^2 + d_t^2 - 2\xi\sqrt{x^2 + d_t^2} + \xi^2 = x^2 - 2xd_r + d_r^2 + d_t^2$$

$$-2\xi\sqrt{x^2 + d_t^2} = -2xd_r + d_r^2 - \xi^2$$

Square both sides

$$4\xi^2(x^2 + d_t^2) = 4x^2 d_r^2 - 4xd_r(d_r^2 - \xi^2) + (d_r^2 - \xi^2)^2$$

$$0 = -4\xi^2 x^2 - 4\xi^2 d_t^2 + 4x^2 d_r^2 - 4xd_r(d_r^2 - \xi^2) + (d_r^2 - \xi^2)^2$$

$$0 = x^2 4(d_r^2 - \xi^2) - 4xd_r(d_r^2 - \xi^2) + (d_r^2 - \xi^2)^2 - 4\xi^2 d_t^2$$

Quadratic eqn: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{4d_r(d_r^2 - \xi^2) \pm \sqrt{4^2 d_r^2 (d_r^2 - \xi^2)^2 - 4^2 (d_r^2 - \xi^2)\left[(d_r^2 - \xi^2)^2 - 4\xi^2 d_t^2\right]}}{8(d_r^2 - \xi^2)}$$

P-2.21 (SP-1) = P-2.28 (DSP-2)

$$= \frac{d_r}{2} \pm \frac{\sqrt{4^2 d_r^2 (d_r^2 - \xi^2)^2 - 4^2 (d_r^2 - \xi^2)^3 + 4^3 (d_r^2 - \xi^2) \xi^2 d_t^2}}{8 (d_r^2 - \xi^2)}$$

$$= \frac{d_r}{2} \pm \frac{\sqrt{4^2 (d_r^2 - \xi^2)^2 \left[d_r^2 - (d_r^2 - \xi^2) \right] + 4^3 (d_r^2 - \xi^2) \xi^2 d_t^2}}{8 (d_r^2 - \xi^2)}$$

$$= \frac{d_r}{2} \pm \frac{\sqrt{4^2 (d_r^2 - \xi^2)^2 \xi^2 + 4^3 (d_r^2 - \xi^2) \xi^2 d_t^2}}{8 (d_r^2 - \xi^2)}$$

$$= \frac{d_r}{2} \pm \frac{\sqrt{4^2 (d_r^2 - \xi^2) \xi^2 \left[(d_r^2 - \xi^2) + 4 d_t^2 \right]}}{8 (d_r^2 - \xi^2)}$$

$$= \frac{d_r}{2} \pm \frac{4 \xi \sqrt{(d_r^2 - \xi^2)(d_r^2 + 4 d_t^2 - \xi^2)}}{8 (d_r^2 - \xi^2)}$$

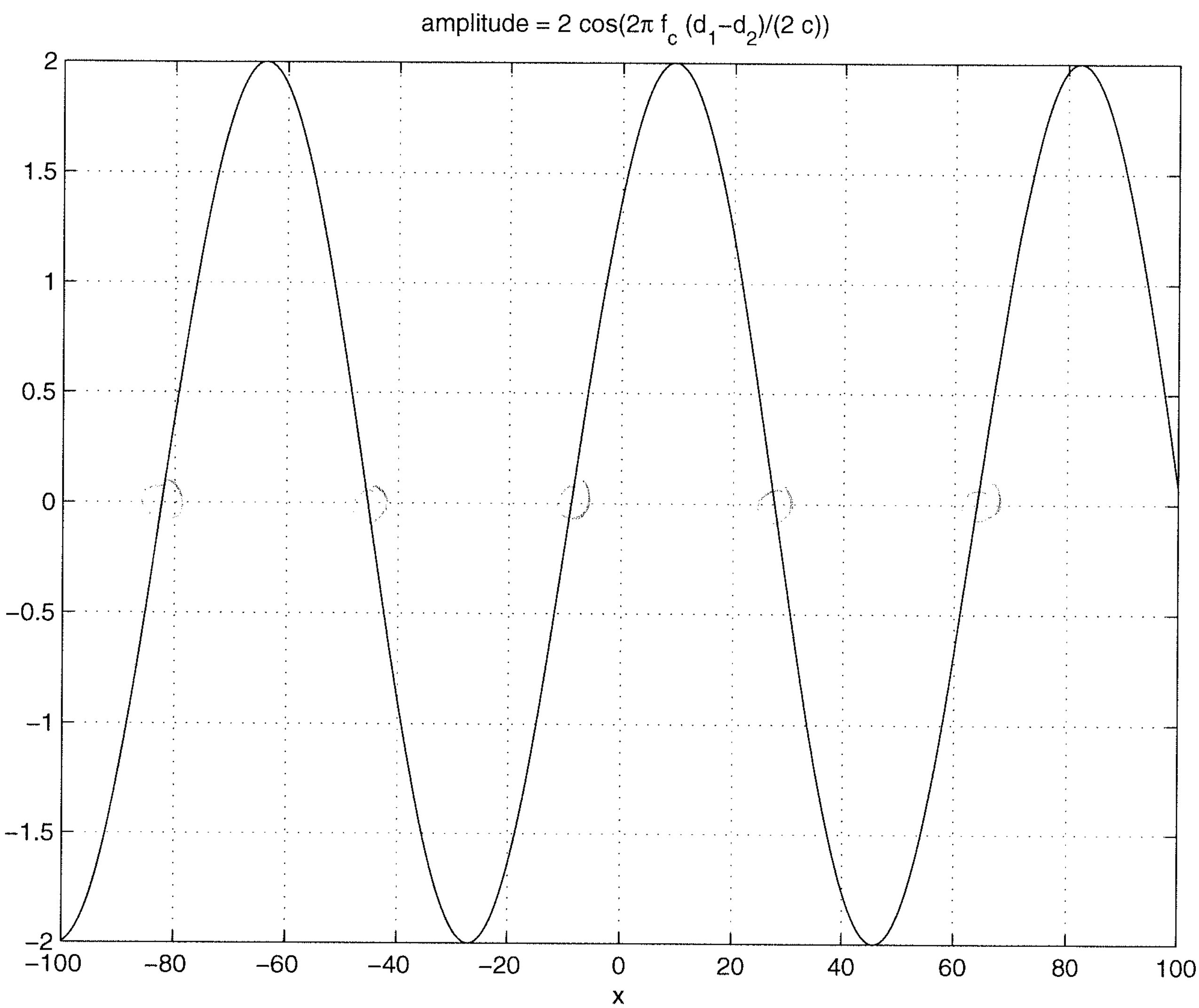
$$= \frac{d_r}{2} \pm \frac{\xi}{2} \sqrt{\frac{d_r^2 + 4 d_t^2 - \xi^2}{d_r^2 - \xi^2}}$$

(d)

```
diary P_2_21_V2.diary

fc=150*10^6;
dt=1000;
dr=55;
c=3*10^8;
fprintf(1,'fc %g dt %g dr %g c %g\n',fc,dt,dr,c);

%make a plot of the amplitude
x=[-100.0 : .1 : 100.0];
d1=sqrt(x.^2 + dt^2);
d2=dr+sqrt((x-dr).^2 + dt^2);
argument=2.0*pi*fc*(d1-d2)/(2*c);
amplitude=2*cos(argument);
figure;
plot(x,amplitude);
title('amplitude = 2 cos(2\pi f_c (d_1-d_2)/(2 c))');
xlabel('x');
grid;
print -deps P_2_21_V2.a.eps

% %make a plot of the argument
% figure;
% plot(x,argument);
% title('argument');
% xlabel('x');
% grid;
% print -deps P_2_21_V2.b.eps
%
% %make a plot of the argument/pi - 0.5
% figure;
% plot(x,argument/pi - 0.5);
% title('argument/\pi - 0.5');
% xlabel('x');
% grid;
% print -deps P_2_21_V2.c.eps
% %x such that argument/pi - 0.5 is an integer are x where the amplitude is zero.

m=[-1:-1:-50];

%evaluate xi
xi=(0.5+m).*c./fc + dr;

%evaluate the solution of the quadratic equation
tmp1=dr/2;
tmp2=0.5.*xi.*sqrt( (dr^2 + 4*dt^2 - xi.^2)./(dr^2 - xi.^2) );
xplus2=tmp1+tmp2;
xminus2=tmp1-tmp2;

%test essentially the first equation
testxplus2=sqrt(xplus2.^2 + dt^2) - xi - sqrt( (xplus2 - dr).^2 + dt^2 );
testxminus2=sqrt(xminus2.^2 + dt^2) - xi - sqrt( (xminus2 - dr).^2 + dt^2 );

for mm=1:length(m)
  fprintf(1,'m %d',m(mm))
  if isreal(xplus2(mm)) & abs(testxplus2(mm))<sqrt(eps)
    fprintf(1,' xplus2 %g',xplus2(mm))
  end
  if isreal(xminus2(mm)) & abs(testxminus2(mm))<sqrt(eps)
    fprintf(1,' xminus2 %g',xminus2(mm))
  end
  fprintf(1,'\n');
end

diary off
```

```
fc 1.5e+08 dt 1000 dr 55 c 3e+08
m -1 xplus2 5199.83
m -2 xplus2 2929.97
m -3 xplus2 2209.82
m -4 xplus2 1815.27
m -5 xplus2 1553.4
m -6 xplus2 1361.01
m -7 xplus2 1210.43
m -8 xplus2 1087.31
m -9 xplus2 983.38
m -10 xplus2 893.462
m -11 xplus2 814.139
m -12 xplus2 743.042
m -13 xplus2 678.464
m -14 xplus2 619.142
m -15 xplus2 564.111
m -16 xplus2 512.621
m -17 xplus2 464.074
m -18 xplus2 417.988
m -19 xplus2 373.963
m -20 xplus2 331.665
m -21 xplus2 290.809
m -22 xplus2 251.148
m -23 xplus2 212.468
m -24 xplus2 174.572
m -25 xplus2 137.287
m -26 xplus2 100.448
m -27 xplus2 63.9014
m -28 xplus2 27.5 xminus2 27.5
m -29 xplus2 -8.90144
m -30 xplus2 -45.4478
m -31 xplus2 -82.2869
m -32 xplus2 -119.572
m -33 xplus2 -157.468
m -34 xplus2 -196.148
m -35 xplus2 -235.809
m -36 xplus2 -276.665
m -37 xplus2 -318.963
m -38 xplus2 -362.988
m -39 xplus2 -409.074
m -40 xplus2 -457.621
m -41 xplus2 -509.111
m -42 xplus2 -564.142
m -43 xplus2 -623.464
m -44 xplus2 -688.042
m -45 xplus2 -759.139
m -46 xplus2 -838.462
m -47 xplus2 -928.38
m -48 xplus2 -1032.31
m -49 xplus2 -1155.43
m -50 xplus2 -1306.01
```

this are the zeros that you can see on the plot.



amplitude = 2 cos(2π f_c (d_1−d_2)/(2 c))
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−100
−80
−60
−40
−20
0
20
40
60
80
100
x