ECE 2200 / Spring 2016 / Problem Set 10 / Solutions

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(a) $\tilde{x}_1[n] \quad n \in \mathbb{Z}, \quad \tilde{x}_1[n+N] = \tilde{x}_1[n]$

$$\tilde{x}_2[n] = \tilde{x}_1[n-n_0]$$

$$\tilde{X}_1[k] = \sum_{n=0}^{N-1} \tilde{x}_1[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$\tilde{X}_2[k] = \sum_{n=0}^{N-1} \tilde{x}_2[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} \tilde{x}_1[n-n_0]\, e^{-j\frac{2\pi}{N}nk}$$

$$m = n - n_0 \qquad \begin{cases} n = 0 \Rightarrow m = -n_0 \\ n = N-1 \Rightarrow m = N-1-n_0 \end{cases}$$
$$\Rightarrow n = m + n_0$$

$$= \sum_{m=-n_0}^{N-1-n_0} \tilde{x}_1[m]\, e^{-j\frac{2\pi}{N}(m+n_0)k}$$

$$= e^{-j\frac{2\pi}{N}n_0 k} \sum_{m=-n_0}^{N-1-n_0} \tilde{x}_1[m]\, e^{-j\frac{2\pi}{N}mk}$$

use $n_0 > 0$

$$= e^{-j\frac{2\pi}{N}n_0 k}\left[\underbrace{\sum_{m=-n_0}^{-1} \tilde{x}_1[m]\, e^{-j\frac{2\pi}{N}mk}} + \sum_{m=0}^{N-1-n_0} \tilde{x}_1[m]\, e^{-j\frac{2\pi}{N}mk}\right]$$

$$l = m + N \qquad \begin{cases} m = -n_0 \Rightarrow l = -n_0 + N \\ m = -1 \Rightarrow l = -1 + N \end{cases}$$
$$\Rightarrow m = l - N$$

$$= e^{-j\frac{2\pi}{N}n_0 k}\left[\sum_{l=N-n_0}^{N-1} \underbrace{\tilde{x}_1[l-N]}_{=\tilde{x}_1[l] \text{ by periodicity of } x_1} \underbrace{e^{-j\frac{2\pi}{N}(l-N)k}}_{= e^{-j\frac{2\pi}{N}lk} \text{ by } e^{-j2\pi M} = 1 \text{ for } M \in \mathbb{Z}} + \sum_{m=0}^{N-1-n_0} \tilde{x}_1[m]\, e^{-j\frac{2\pi}{N}mk}\right]$$

$$= e^{-j\frac{2\pi}{N}n_0 k}\left[\sum_{l=N-n_0}^{N-1} \tilde{x}_1[l]\, e^{-j\frac{2\pi}{N}lk} + \sum_{m=0}^{N-1-n_0} \tilde{x}[m]\, e^{-j\frac{2\pi}{N}mk}\right]$$

$$= e^{-j\frac{2\pi}{N}n_0 k} \sum_{m=0}^{N-1} \tilde{x}_1[m]\, e^{-j\frac{2\pi}{N}mk}$$

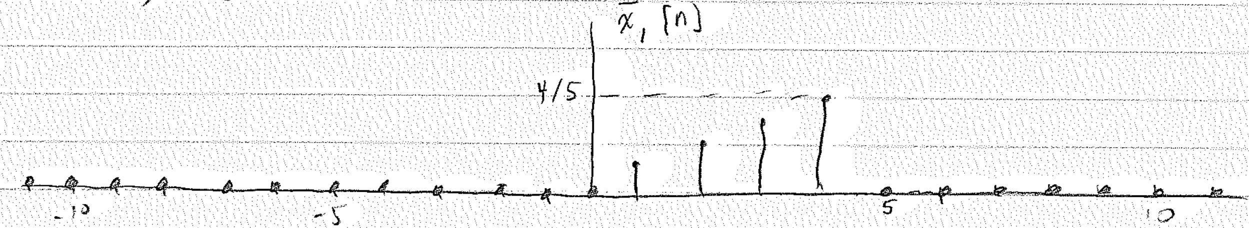
$$= e^{-j\frac{2\pi}{N}n k}\, \tilde{X}_1[k]$$

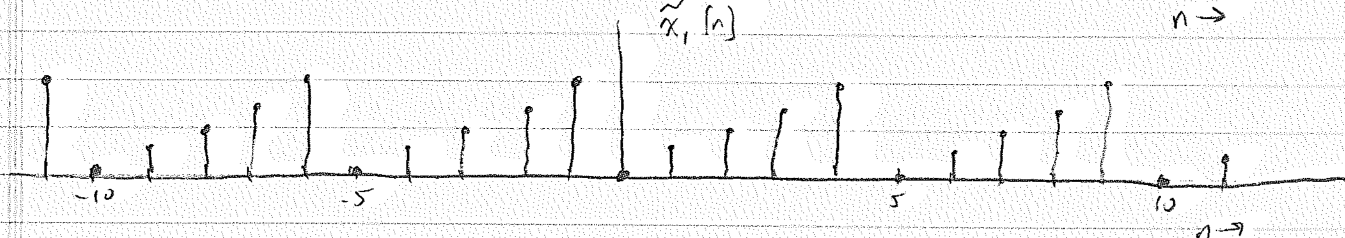
(b)

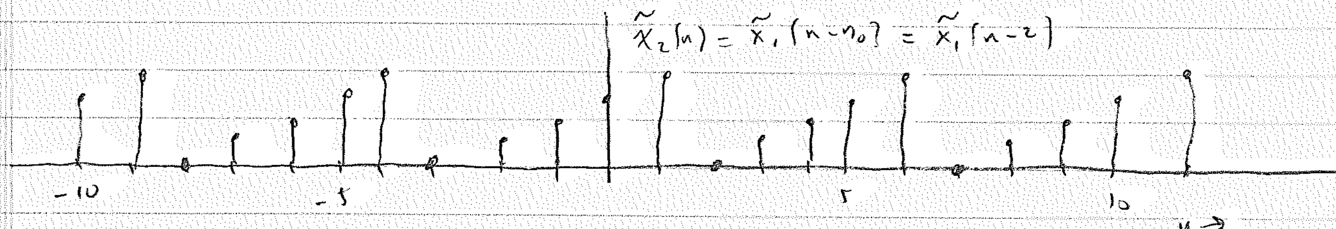
$$\bar{x}_1[n] = \begin{cases} n/N & n \in \{0, 1, \dots, N-1\} \\ 0 & \end{cases}$$

$$\tilde{x}_1[n] = \sum_{\ell=-\infty}^{+\infty} \bar{x}_1[n - \ell N]$$

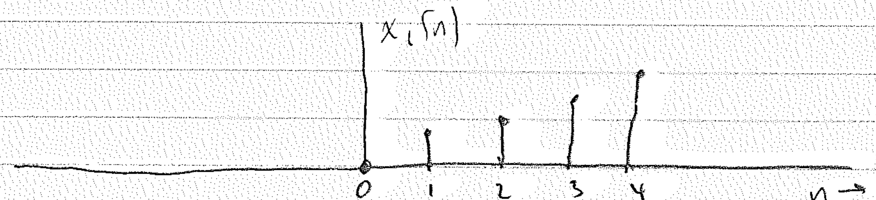
$N = 5, \ n_0 = 2$

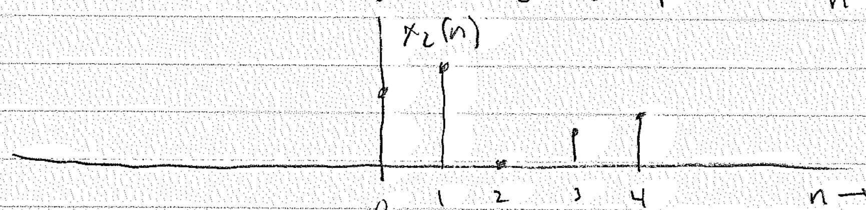






(c)





(d) $\tilde{x}_1[n] \longleftrightarrow X_1[k]$, $\tilde{x}_2[n] \longleftrightarrow X_2[k]$

$$\tilde{y}[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\, \tilde{x}_2[n-m]$$

$$Y[k] = \sum_{n=0}^{N-1} \tilde{y}[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} \tilde{x}_1[m]\, \tilde{x}_2[n-m] \right] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \left[\sum_{n=0}^{N-1} \tilde{x}_2[n-m]\, e^{-j\frac{2\pi}{N}nk} \right]$$

$$l = n-m \quad \Rightarrow n = l+m \qquad \begin{cases} n=0 \Rightarrow l = -m \\ n = N-1 \Rightarrow l = N-1-m \end{cases}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \left[\sum_{l=-m}^{N-1-m} \tilde{x}_2[l]\, e^{-j\frac{2\pi}{N}(l+m)k} \right]$$

$$= \underbrace{\left[\sum_{m=0}^{N-1} \tilde{x}_1[m]\, e^{-j\frac{2\pi}{N}mk} \right]}_{X_1[k]} \underbrace{\left[\sum_{l=-m}^{N-1-m} \tilde{x}_2[l]\, e^{-j\frac{2\pi}{N}lk} \right]}$$

$m \geq 0$ always

$$= \sum_{l=-m}^{-1} \tilde{x}_2[l]\, e^{-j\frac{2\pi}{N}lk} + \sum_{l=0}^{N-1-m} \tilde{x}_2[l]\, e^{-j\frac{2\pi}{N}lk}$$

$$\xi = l+N \qquad \begin{cases} l=-m \Rightarrow \xi = -m+N \\ l = -1 \Rightarrow \xi = -1+N \end{cases}$$
$$l = \xi - N$$

$$= \sum_{\xi=N-m}^{N-1} \underbrace{\tilde{x}_2[\xi-N]}_{=\, x_2[\xi] \text{ by periodicity}}\, \underbrace{e^{-j\frac{2\pi}{N}(\xi-N)k}}_{=\, e^{-j\frac{2\pi}{N}\xi k} \text{ because } e^{-j2\pi M} = 1 \text{ for } M \in \mathbb{Z}} + \sum_{l=0}^{N-1-m} \tilde{x}_2[l]\, e^{-j\frac{2\pi}{N}lk}$$

$$= \sum_{\xi=N-m}^{N-1} \tilde{x}_2[\xi]\, e^{-j\frac{2\pi}{N}\xi k} + \sum_{l=0}^{N-1-m} \tilde{x}_2[l]\, e^{-j\frac{2\pi}{N}lk}$$

$$= \sum_{l=0}^{N-1} \tilde{x}_2[l]\, e^{-j\frac{2\pi}{N}lk} = X_2[k]$$

So $Y[k] = X_1[k]\, X_2[k]$.

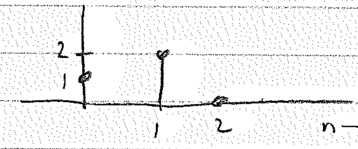
(e) The ideas of (d) will not work. One way in which to see that there is a problem is to note that (d) provides an answer that has only $N-1$ different values as $n \in \{0, \dots, N-1\}$. But the regular convolution of two sequences each of duration N gives an answer that is $2N-1$ in duration.

The previous observation provide a hint to a solution. Consider adding zeros to each of x_1 and x_2 so that they are each $2N-1$ long — called zero padding.

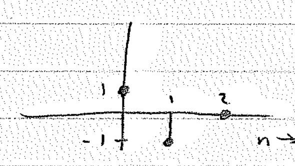
Example:

$$x_1(n) = \begin{cases} 1 & n=0 \\ 2 & n=1 \end{cases} \qquad x_2(n) = \begin{cases} 1 & n=0 \\ -1 & n=1 \end{cases}$$

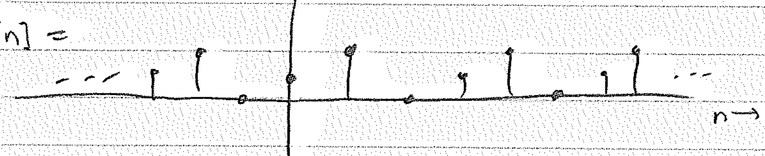
$x_1^{zp}(n) =$



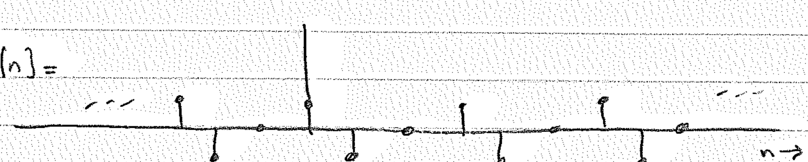
$x_2^{zp}(n) =$



$\tilde{x}_1^{zp}[n] =$

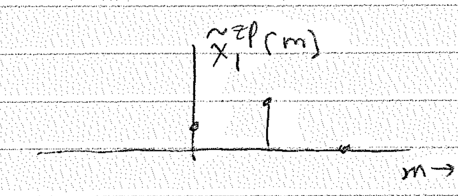


$\tilde{x}_2^{zp}[n] =$

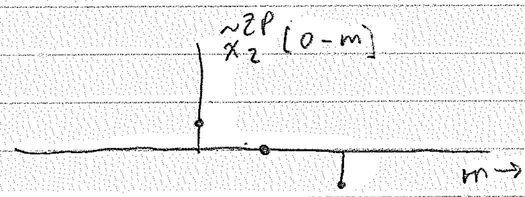


$$\tilde{y}^{zp}[n] = \sum_{m=0}^{N-1} \tilde{x}_1^{zp}[m]\, \tilde{x}_2^{zp}[n-m]$$

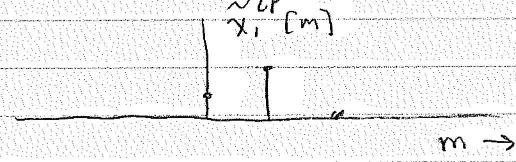
$n=0$



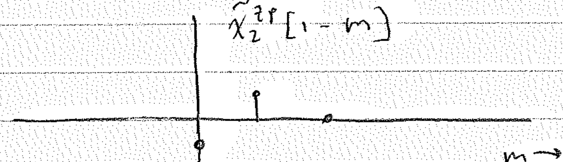
$$\tilde{y}^{zp}[0] = (1)(1) + (2)(0) + (0)(-1) = 1$$



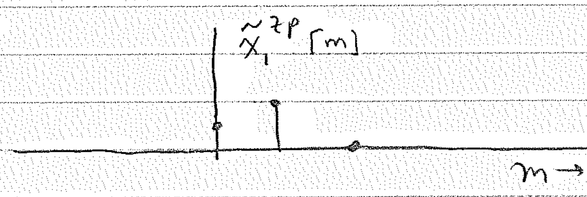
$n=1$



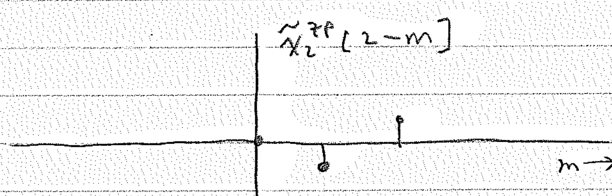
$$\tilde{y}^{zp}[1] = (1)(-1) + (2)(1) + (0)(0) = 1$$



$n=2$

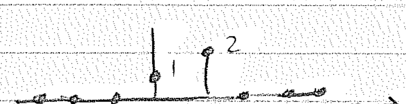


$$\tilde{y}^{zp}[2] = (1)(0) + (2)(-1) + (0)(1) = -2$$

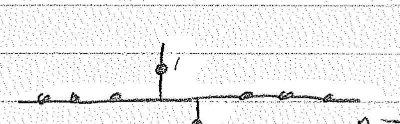


Now do the calculation using

$$x_1^{\infty}[n] = \begin{cases} x_1[n] & n \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} =$$



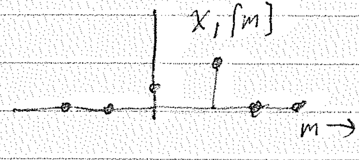
$$x_2^{\infty}[n] = \begin{cases} x_2[n] & n \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} =$$



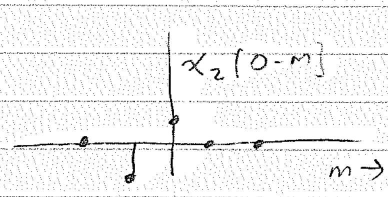
$$y^{\infty}[n] = \sum_{m=-\infty}^{+\infty} x_1^{\infty}[m]\, x_2^{\infty}[n-m]$$

$n<0$ no overlap

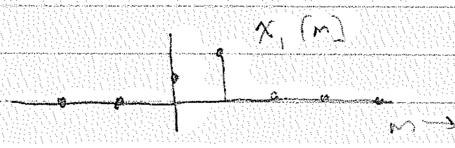
$n=0$



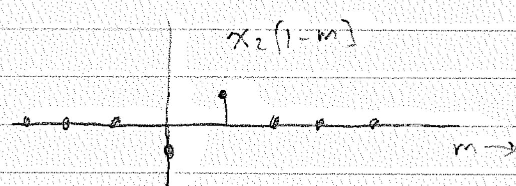
$y^{\infty}[0] = (1)(1) = 1$



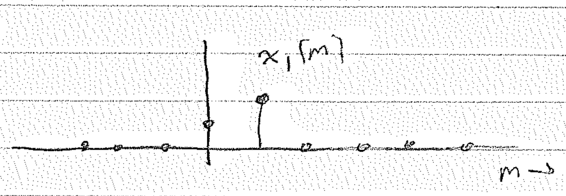
$n=1$



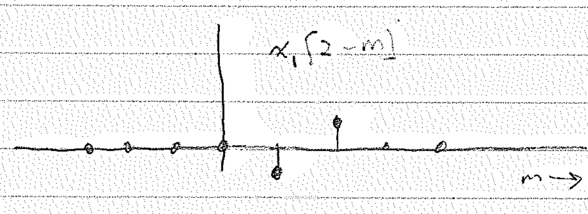
$y^{\infty}[1] = (1)(-1) + (2)(1) = 1$



$n=2$



$y^{\infty}[2] = (2)(-1) = -2$



$n>2$ no overlap

Note that $y^{\infty}[n] = \tilde{y}^{zp}[n]$ on $n \in \{0,1,2\} = \{0,\dots,N-1\}$.

Therefore, this example works

The zero padding approach works in general because the part of the circular shift that you do not want is always multiplied by zeros from the zero padding.

$$x[n] = e^{j(\Omega_0 n + \phi_0)} \qquad n \in \{0, 1, \ldots, N-1\}$$

(a) $$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} e^{j(\Omega_0 n + \phi_0)} e^{-j\frac{2\pi}{N}nk}$$

$$= e^{j\phi_0} \sum_{n=0}^{N-1} e^{j(\Omega_0 - \frac{2\pi}{N}k)n}$$

$$= e^{j\phi_0} \sum_{n=0}^{N-1} \left[e^{j(\Omega_0 - \frac{2\pi}{N}k)}\right]^n$$

$$= e^{j\phi_0} \frac{1 - e^{j(\Omega_0 - \frac{2\pi}{N}k)N}}{1 - e^{j(\Omega_0 - \frac{2\pi}{N}k)}}$$

I could use $e^{-j\frac{2\pi}{N}kN} = 1$ but then I would get a less symmetric final answer

$$= e^{j\phi_0} \frac{e^{j(\Omega_0 - \frac{2\pi}{N}k)\frac{N}{2}}\left[e^{-j(\Omega_0 - \frac{2\pi}{N}k)\frac{N}{2}} - e^{+j(\Omega_0 - \frac{2\pi}{N}k)\frac{N}{2}}\right]}{e^{j(\Omega_0 - \frac{2\pi}{N}k)\frac{1}{2}}\left[e^{-j(\Omega_0 - \frac{2\pi}{N}k)\frac{1}{2}} - e^{+j(\Omega_0 - \frac{2\pi}{N}k)\frac{1}{2}}\right]}$$

$$= e^{j\phi_0} e^{j\left[(\Omega_0 - \frac{2\pi}{N}k)\frac{N}{2} - (\Omega_0 - \frac{2\pi}{N}k)\frac{1}{2}\right]} \frac{-2j\sin(\Omega_0 - \frac{2\pi}{N}k)\frac{N}{2}}{-2j\sin(\Omega_0 - \frac{2\pi}{N}k)\frac{1}{2}}$$

$$= e^{j\phi_0} e^{j(\Omega_0 - \frac{2\pi}{N}k)\frac{N-1}{2}} \frac{\sin(\Omega_0 - \frac{2\pi}{N}k)\frac{N}{2}}{\sin(\Omega_0 - \frac{2\pi}{N}k)\frac{1}{2}}$$

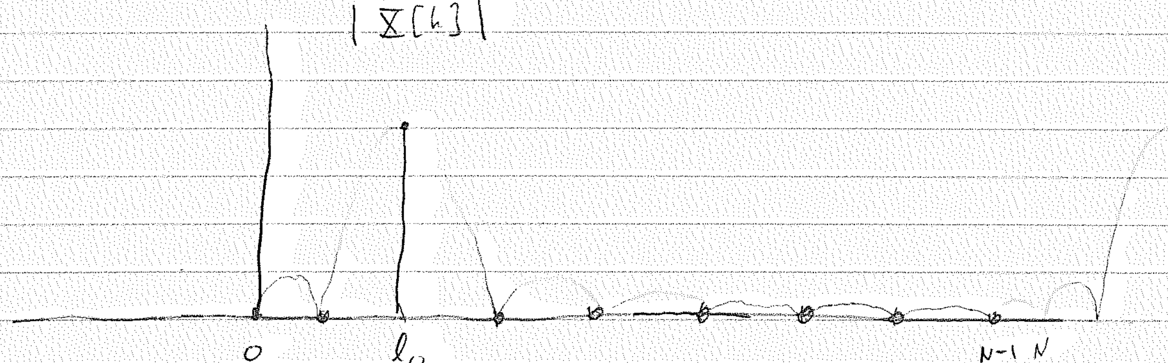
(b) The plot will appear quite different depending on whether or not $\Omega_0 = \frac{2\pi}{N} l_0$ for some $l_0 \in \mathbb{Z}$.

If $\Omega_0 = \frac{2\pi}{N} l_0$ for some $l_0 \in \mathbb{Z}$

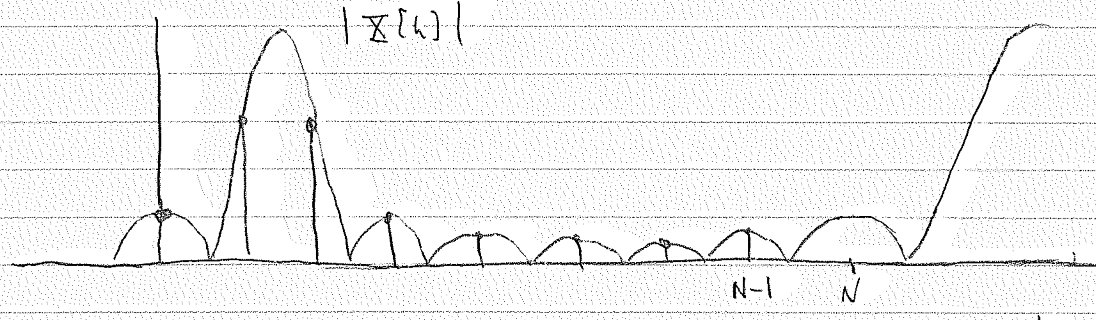
$$\frac{\sin\left(\Omega_0 - \frac{2\pi}{N}k\right)\frac{N}{2}}{\sin\left(\Omega_0 - \frac{2\pi}{N}k\right)\frac{1}{2}} = \frac{\sin \frac{2\pi}{N}(l_0 - k)\frac{N}{2}}{\sin \frac{2\pi}{N}(l_0 - k)\frac{1}{2}}$$

When $k \neq l_0$ the the denominator is finite but the numerator is 0 so the ratio is 0

when $k = l$ both the numerator and denominator are zero and the limit is N.



If $\Omega_0 \neq \frac{2\pi}{N} l_0$ for any $l_0 \in \mathbb{Z}$ then the ratio of sine functions is not zero



(c) If you can control N then the condition for achieving $\Omega_0 = \frac{2\pi}{N} l_0$ $(l_0 \in \mathbb{Z})$ is that $\Omega_0/2\pi$ is rational.

$$x[n] = \begin{cases} 1 & n \in \{0, 1, \ldots, M-1\} \\ 0 & n \in \{M, \ldots, N-1\} \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{M-1} e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{M-1} \left(e^{-j\frac{2\pi}{N}k}\right)^n$$

$$= \frac{1 - e^{-j\frac{2\pi}{N}kM}}{1 - e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{e^{-j\frac{2\pi}{N}k\frac{M}{2}}\left(e^{+j\frac{2\pi}{N}k\frac{M}{2}} - e^{-j\frac{2\pi}{N}k\frac{M}{2}}\right)}{e^{-j\frac{2\pi}{N}\frac{k}{2}}\left(e^{+j\frac{2\pi}{N}\frac{k}{2}} - e^{-j\frac{2\pi}{N}\frac{k}{2}}\right)}$$

$$= e^{-j\frac{2\pi}{N}k\left(\frac{M}{2}-\frac{1}{2}\right)} \frac{2j \sin \frac{2\pi}{N}k\frac{M}{2}}{2j \sin \frac{2\pi}{N}\frac{k}{2}}$$

$$= e^{-j\frac{2\pi}{N}\frac{M-1}{2}k} \frac{\sin \frac{2\pi}{N}\frac{k}{2}M}{\sin \frac{2\pi}{N}\frac{k}{2}}$$

if $M=1$ then $x[n] = \delta[n]$ and $X[k] = 1$