ECE 2200 and ENGRD 2220
Signals and Systems
Spring 2016
Problem Set 10—The Last Problem Set for ECE 2200 and ENGRD 2220!
Due May 6, 2016 at 5:00PM.
Location to turn in: "ECE 2200" box on Phillips Hall 2nd floor

1. The Discrete Fourier Transform (DFT) pair of synthesis and analysis equations is

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi}{N}nk\right) \tag{119}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \exp\left(j\frac{2\pi}{N}nk\right). \tag{120}$$

Please note that Eqs. 119 and 120 imply that $X[k+N] = X[k]$ and $x[n+N] = x[n]$ so both are periodic.

There are two points of view:

(a) The sequence $x[n]$ has exactly N elements indexed by $n \in \{0, 1, \ldots, N-1\}$. This is all you need in order to evaluate the right hand side of Eq. 119.

(b) The sequence $x[n]$ is N elements ($n \in \{0, 1, \ldots, N-1\}$) extracted from an infinite periodic sequence $\tilde{x}[n]$ with period N. Note that this is consistent with the periodicity of Eq. 120 if you happen to evaluate Eq. 120 for $n \notin \{0, 1, \ldots, N-1\}$. Notation: A variable with a tilde will be a periodic extension of a finite length sequence having the same name without the tilde, i.e., $\tilde{x}[n]$ and $x[n]$.

Questions:

(a) Suppose you take the periodic point of view. Suppose you have an infinite sequence $\tilde{x}_1[n]$ that is periodic with period N and has DFT $X_1[k]$. Suppose you define $\tilde{x}_2[n] = \tilde{x}_1[n-n_0]$ (n_0 is an integer) which is a second infinite sequence that is periodic with period N. What is the relationship between the DFT of $\tilde{x}_2[n]$, denoted by $X_2[k]$, and $X_1[k]$? Consider only the case of $n_0 \geq 0$, the case of $n_0 < 0$ is fundamentally the same.

(b) Define a particular $\tilde{x}_1[n]$ by giving one period:

$$\bar{x}_1[n] = \begin{cases} n/N, & n \in \{0, 1, \ldots, N-1\} \\ 0, & \text{otherwise} \end{cases} \tag{121}$$

$$\tilde{x}_1[n] = \sum_{l=-\infty}^{+\infty} \bar{x}_1[n-lN]. \tag{122}$$

I use $\bar{x}_1[n]$ rather than $x_1[n]$ because $\bar{x}_1[n]$ is defined for all values of n since otherwise Eq. 122 does not make any sense. However, $x_1[n]$ need not be defined for all values of n. For the case $N = 5$ and $n_0 = 2$, please plot $\bar{x}[n]$, $\tilde{x}_1[n]$, and the corresponding $\tilde{x}_2[n]$.

(c) Now, suppose you change to the point of view that a sequence has exactly N elements indexed by $\{0, 1, \ldots, N-1\}$. These N elements are exactly the values of the periodic sequence on the interval $n \in \{0, 1, \ldots, N-1\}$. For the sequence defined in Part 1b with $N = 5$ and $n_0 = 2$, please plot $x_1[n]$ and $x_2[n]$. The relationship between $x_2[n]$ and $x_1[n]$ is not what a person would typically call "translation" or "delay" or "shift" because whatever is "pushed out" of the left or right side "wraps around" and appears on the right or left side. This sort of relationship is called "circular".

(d) Return to the periodic point of view. Suppose you have two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ and you define the convolution of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ with result $\tilde{y}[n]$ by

$$\tilde{y}[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]. \tag{123}$$

If $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ have DFTs $X_1[k]$ and $X_2[k]$, then what is the DFT of $\tilde{y}[n]$, denoted by $Y[k]$?

(e) In many applications, what you want is to convolve two finite duration signals. Specifically, let $x_1[n]$ be defined for $\{0, 1, \ldots, N-1\}$, let $x_2[n]$ be defined for $\{0, 1, \ldots, N-1\}$, and define

$$x_1^\infty[n] = \begin{cases} x_1[n], & n \in \{0, 1, \ldots, N-1\} \\ 0, & \text{otherwise} \end{cases} \tag{124}$$

$$x_2^\infty[n] = \begin{cases} x_2[n], & n \in \{0, 1, \ldots, N-1\} \\ 0, & \text{otherwise} \end{cases}. \tag{125}$$

The goal is to compute our usual definition of $y[n] = x_1^\infty[n] * x_2^\infty[n]$, i.e.,

$$y[n] = \sum_{m=-\infty}^{+\infty} x_1^\infty[m] x_2^\infty[n-m]. \tag{126}$$

Will the calculations of Part 1d give you what you want? Hint: Because convolution involves "flip and shift", you expect that the answer is no because the "shift" you get in Part 1d is a circular shift but you want the standard shift. Now consider applying Part 1d to what are called "zero-padded" signals,

$$x_1^{\mathrm{zp}}[n] = \begin{cases} x_1[n], & n \in \{0, 1, \ldots, N-1\} \\ 0, & n \in \{N, N+1, \ldots, 2N-2\} \end{cases} \tag{127}$$

$$x_2^{\mathrm{zp}}[n] = \begin{cases} x_2[n], & n \in \{0, 1, \ldots, N-1\} \\ 0, & n \in \{N, N+1, \ldots, 2N-2\} \end{cases}, \tag{128}$$

with corresponding periodic signals $\tilde{x}_1^{\mathrm{zp}}[n]$ and $\tilde{x}_2^{\mathrm{zp}}[n]$ which are periodic with period $2N-1$. (The superscript "zp" stands for "zero padding"). You must do more work because all of the DFTs are computed based on signals that are $2N-1$ samples in duration. Determine if this gives the correct answer by trying with the simple case

$$x_1[n] = \delta[n] + 2\delta[n-1] \tag{129}$$

$$x_2[n] = \delta[n] - \delta[n-1]. \tag{130}$$

2. Consider the sequence

$$x[n] = \exp(j(\Omega_0 n + \phi_0)) \tag{131}$$

defined on the interval $n \in \{0, 1, \ldots, N-1\}$.

(a) Please compute the Discrete Fourier Transform of $x[n]$.

(b) Please sketch the result separately for the case where $\Omega_0 = (2\pi/N)l_0$ for some $l_0 \in \mathcal{Z}$ and when $\Omega_0 \neq (2\pi/N)l_0$ for any $l_0 \in \mathcal{Z}$.

(c) Suppose that you can control the value of N. What is the condition on Ω_0 such that you can write it in the form $\Omega_0 = (2\pi/N)l_0$ for some $l_0 \in \mathcal{Z}$?

3. Consider the sequence

$$x[n] = \begin{cases} 1, & n \in \{0, 1, \ldots, M-1\} \\ 0, & n \in \{M, \ldots, N-1\} \end{cases} \tag{132}$$

defined on the interval $n \in \{0, 1, \ldots, N-1\}$.

(a) Please compute the Discrete Fourier Transform of $x[n]$.

(b) Please plot $x[n]$ and $X[k]$ for the special case $M = 1$.