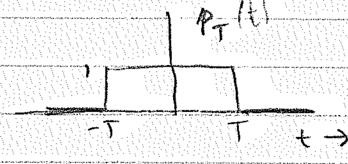
(a) $$\mathcal{U}(\omega) = \int_{-\infty}^{+\infty} u(t) e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{t=0}^{\infty}$$

$$= \, ?$$

(b) $$p_T(t) = \begin{cases} 1 & |t| < T \\ 0 & \text{otherwise} \end{cases} =$$



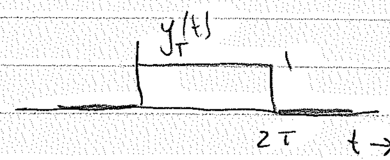

$$\mathrm{sinc}(z) = \frac{\sin \pi z}{\pi z}$$

$$P_T(\omega) = \int_{-\infty}^{+\infty} p_T(t) e^{-j\omega t}\, dt = \int_{-T}^{+T} e^{-j\omega t}\, dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_{t=-T}^{+T}$$

$$= \frac{1}{-j\omega}\left[ e^{-j\omega T} - e^{+j\omega T} \right] = \frac{1}{-j\omega}\left[ -2j \sin \omega T \right] = \frac{1}{\omega} 2 \sin \omega T$$

$$= 2T \frac{\sin(\pi \omega T/\pi)}{\pi \omega T/\pi} = 2T\, \mathrm{sinc}(\omega T/\pi)$$

$$y_T(t) = p_T(t - T) =$$




$$\updownarrow$$

$$Y_T(\omega) = e^{-j\omega T} P_T(\omega)$$

$$u(t) = \lim_{T \to \infty} y_T(t)$$

$$\updownarrow$$

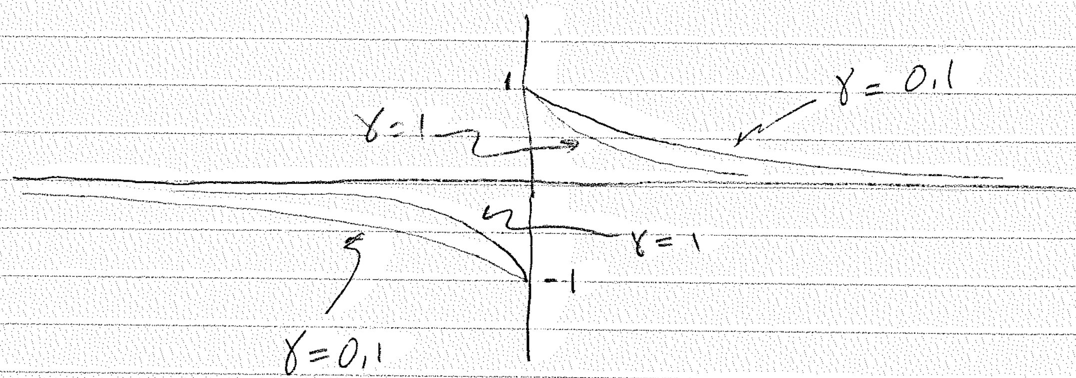
$$\mathcal{U}(\omega) = \mathcal{F}\left\{ \lim_{T\to\infty} y_T(t) \right\} = \lim_{T\to\infty} \mathcal{F}\left\{ y_T(t) \right\} = \lim_{T\to\infty} Y_T(\omega)$$

$$= \lim_{T\to\infty} \left[ e^{-j\omega T} P_T(\omega) \right] = \lim_{T\to\infty} \left[ e^{-j\omega T}\, 2T\, \mathrm{sinc}(\omega T/\pi) \right]$$

$$= \, ?$$

(c)

$$g_\gamma(t) = \begin{cases} -\exp(\gamma t) & t<0 \\ \exp(-\gamma t) & t \geq 0 \end{cases} \qquad \gamma > 0$$




$$G_\gamma(t) = \int_{-\infty}^{+\infty} g_\gamma(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} -e^{\gamma t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-\gamma t} e^{-j\omega t}\, dt$$

$$= -\int_{-\infty}^{0} e^{(\gamma - j\omega)t}\, dt + \int_{0}^{\infty} e^{(-\gamma - j\omega)t}\, dt$$

$$= -\frac{1}{\gamma - j\omega} e^{(\gamma - j\omega)t}\Big|_{t=-\infty}^{0} + \frac{1}{-\gamma - j\omega} e^{(-\gamma - j\omega)t}\Big|_{t=0}^{\infty}$$

$$= -\frac{1}{\gamma - j\omega}\left[1 - 0\right] + \frac{1}{-\gamma - j\omega}\left[0 - 1\right]$$

$$= \frac{1}{-\gamma + j\omega} + \frac{1}{\gamma + j\omega}$$

$$u(t) = \lim_{\gamma \to 0} \frac{1}{2}\left(1 + g_\gamma(t)\right)$$

$$\Downarrow \mathcal{F}$$

$$U(\omega) = \mathcal{F}\left\{\lim_{\gamma \to 0} \frac{1}{2}\left(1 + g_\gamma(t)\right)\right\} = \lim_{\gamma \to 0} \mathcal{F}\left\{\frac{1}{2}\left(1 + g_\gamma(t)\right)\right\}$$

$$= \lim_{\gamma \to 0} \frac{1}{2}\left(2\pi\delta(\omega) + G_\gamma(\omega)\right) = \lim_{\gamma \to 0} \frac{1}{2}\left(2\pi\delta(\omega) + \frac{1}{-\gamma + j\omega} + \frac{1}{\gamma + j\omega}\right)$$

$$= \pi\delta(\omega) + \frac{1}{j\omega} \qquad \text{Success!}$$

(d)

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)\,d\tau$$

$$= \int_{\tau=-\infty}^{+\infty} u(t-\tau)\,x(\tau)\,d\tau$$

$$= u(t) * x(t)$$

Therefore

$$Y(\omega) = U(\omega)\,X(\omega)$$

$$= \left[\frac{1}{j\omega} + \pi\delta(\omega)\right] X(\omega)$$

$$= \frac{1}{j\omega} X(\omega) + \pi\,\delta(\omega)\,X(\omega)$$

$$= \frac{1}{j\omega} X(\omega) + \pi\,X(0)\,\delta(\omega)$$

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(a)

$$x[n] = e^{-\alpha n} \qquad n \in \{0, 1, \dots, N-1\} \qquad \alpha \in \mathbb{R},\ \alpha > 0$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} e^{-\alpha n} e^{-j\frac{2\pi}{N}nk} \qquad\qquad \sum_{n=0}^{N-1} \rho^n = \frac{1-\rho^N}{1-\rho}$$

$$= \sum_{n=0}^{N-1} e^{-(\alpha + j\frac{2\pi}{N}k)n}$$

$$= \frac{1 - e^{-(\alpha + j\frac{2\pi}{N}k)N}}{1 - e^{-(\alpha + j\frac{2\pi}{N}k)}}$$

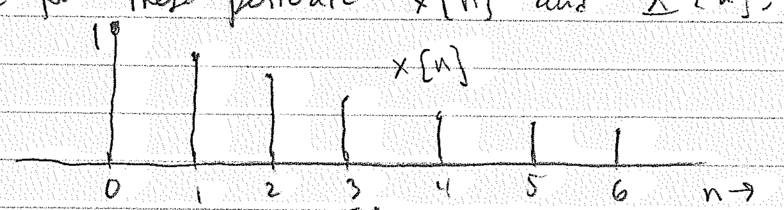
$$= \frac{1 - e^{-\alpha N}\, e^{-j\frac{2\pi}{N}kN}}{1 - e^{-(\alpha + j\frac{2\pi}{N}k)}} \qquad e^{-j\frac{2\pi}{N}kN} = e^{-j2\pi k} = 1$$

$$= \frac{1 - e^{-\alpha N}}{1 - e^{-(\alpha + j\frac{2\pi}{N}k)}}$$

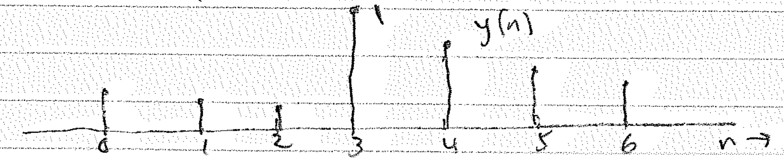
periodic in $k$ with period $N$ as expected.

(b) $Y[k] = \exp\left(-j\frac{2\pi}{N}n_0 k\right) X[k] \longleftrightarrow y[n] = ?$

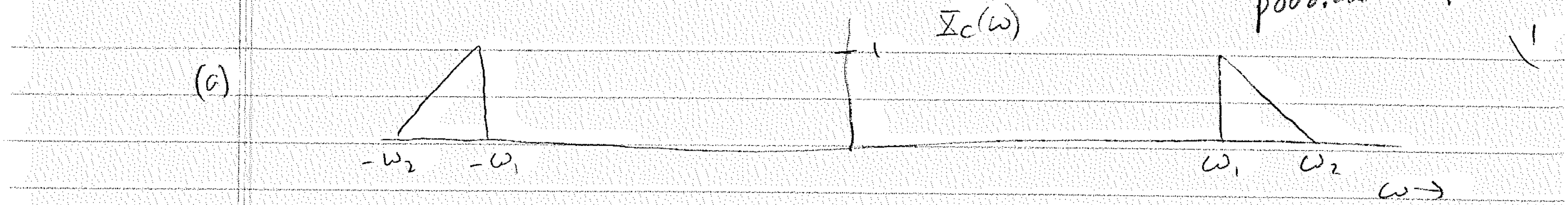
this is the delay thm for DFTs. The key is that delay is cyclic for these periodic $x[n]$ and $X[k]$.

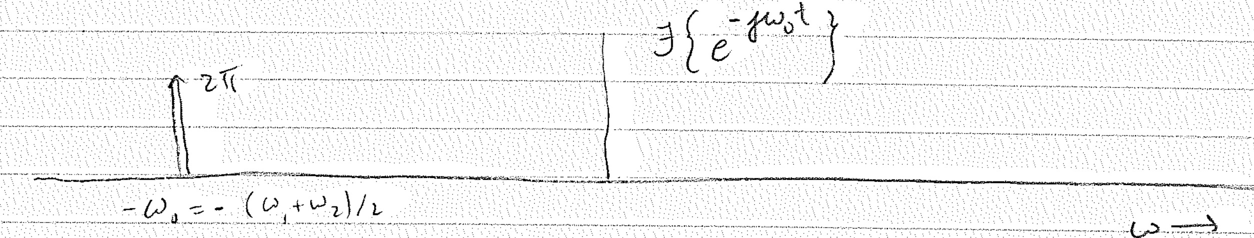


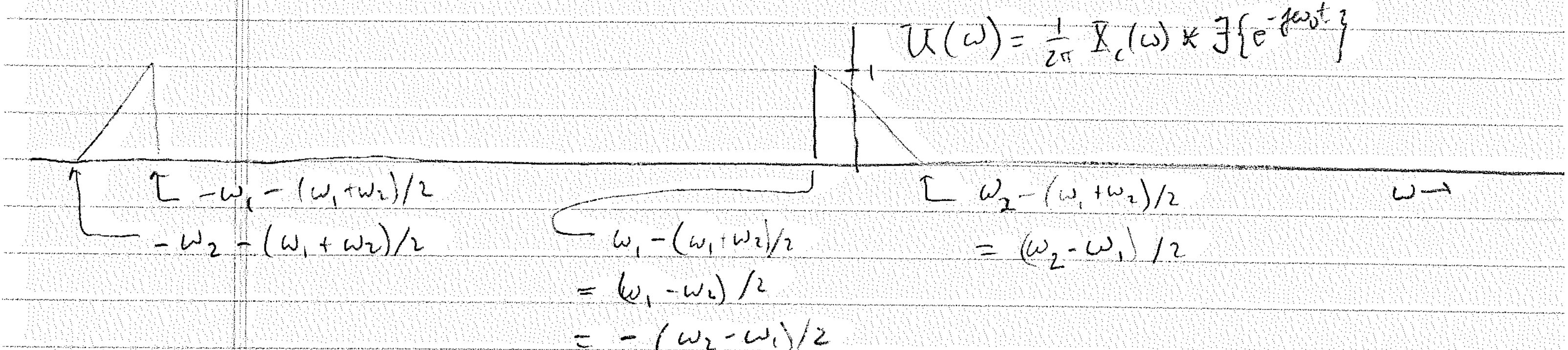

$N = 7$

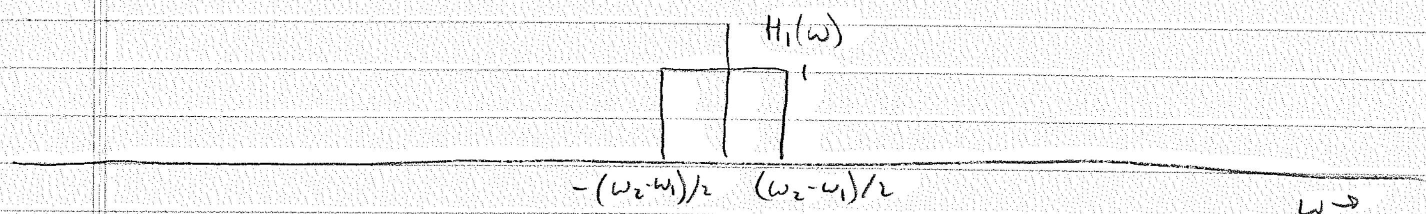


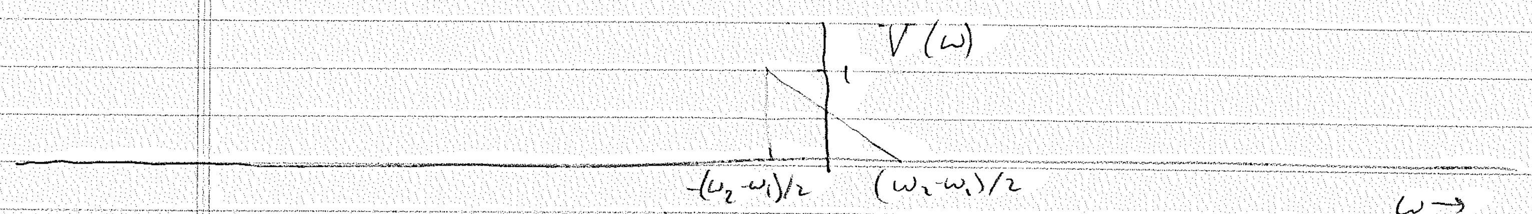
(c)



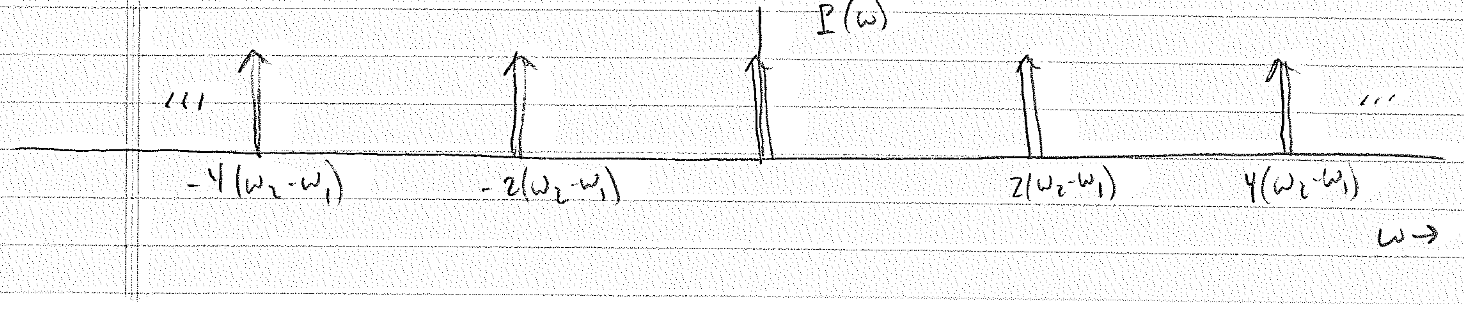




$$p(t) = \sum_{n=-\infty}^{+\infty} \delta\Big(t - n\underbrace{\tfrac{2\pi}{\omega_s}}_{T_s}\Big) \longleftrightarrow P(\omega) = \frac{2\pi}{\underbrace{\left(\tfrac{2\pi}{\omega_s}\right)}_{T_s}} \sum_{k=-\infty}^{+\infty} \delta\Big(\omega - \frac{2\pi k}{\underbrace{\left(\tfrac{2\pi}{\omega_s}\right)}_{T_s}}\Big)$$

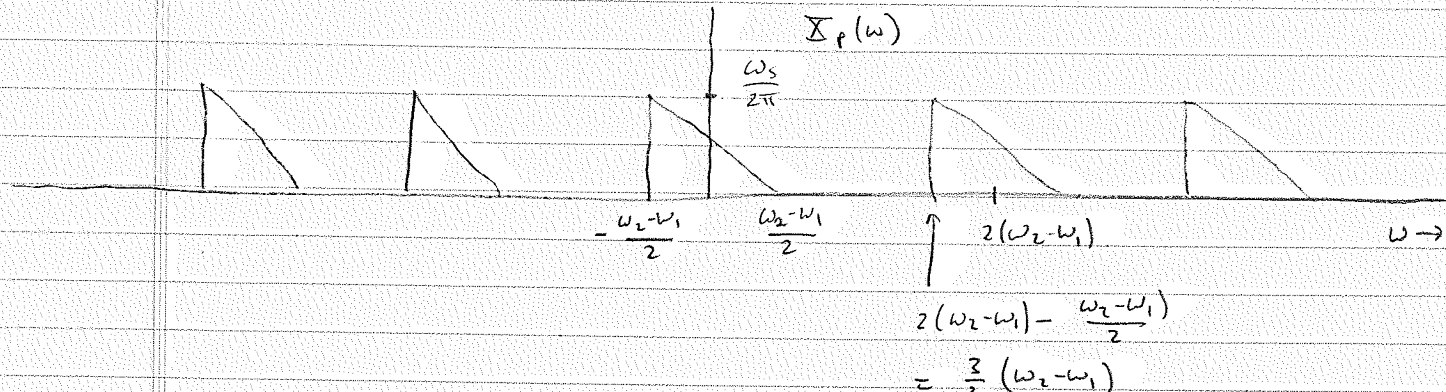
$$\omega_s = 2(\omega_2 - \omega_1)$$

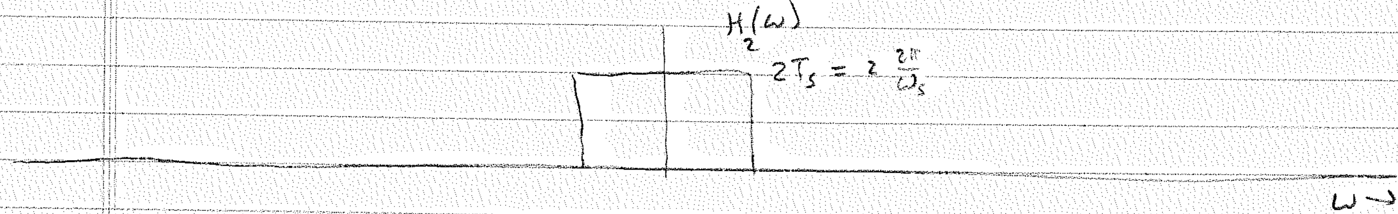
$$= \omega_s \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

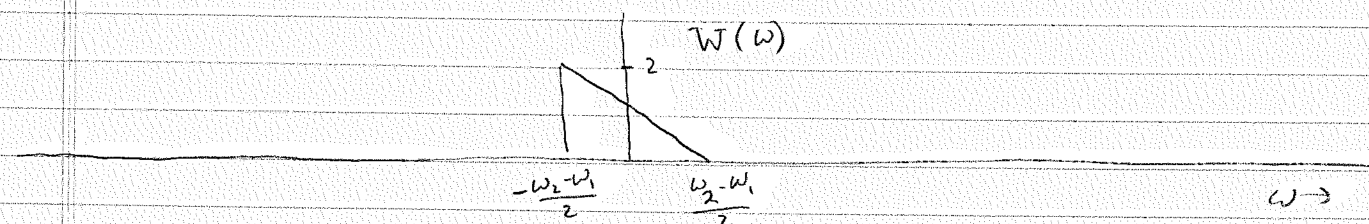


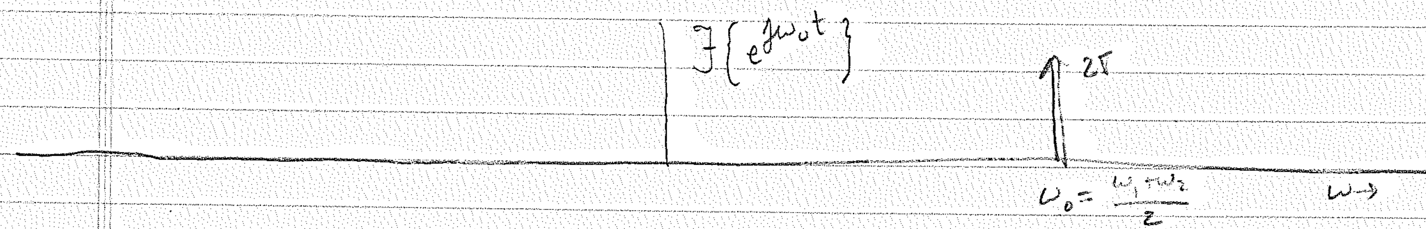
$$x_p(t) = v(t)p(t) \longleftrightarrow X_p(\omega) = \frac{1}{2\pi} V(\omega) * P(\omega)$$

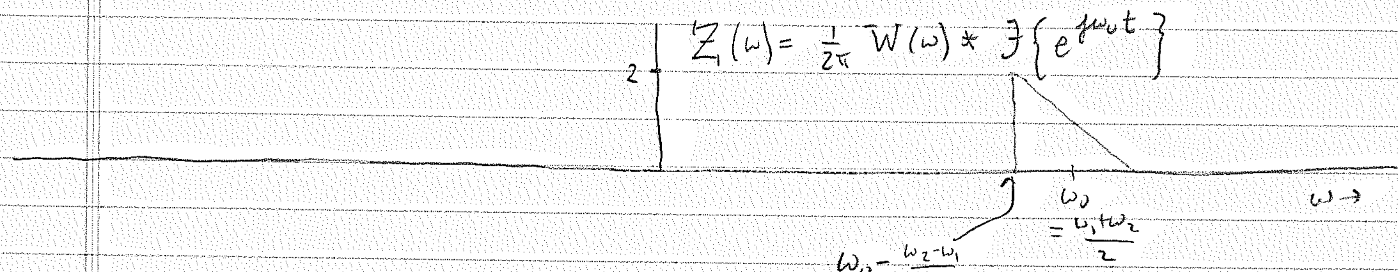
$$= \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{+\infty} V(\omega - k\omega_s)$$



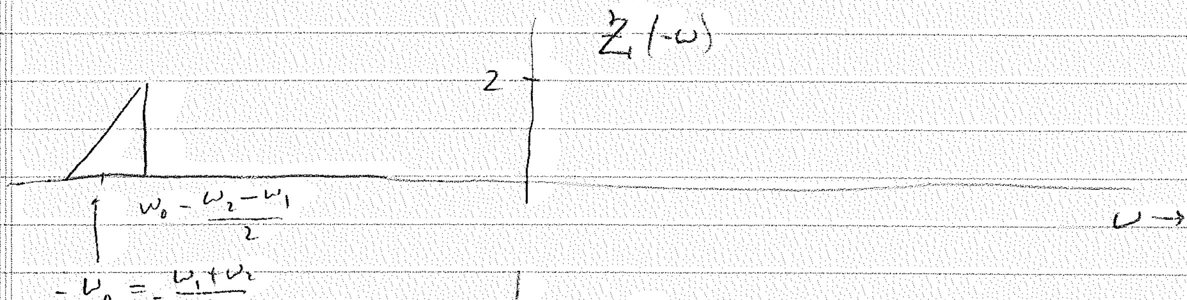
$$x_r(t) = \mathrm{Re}\{z(t)\} = \frac{1}{2}\left[z(t) + z^*(t)\right]$$

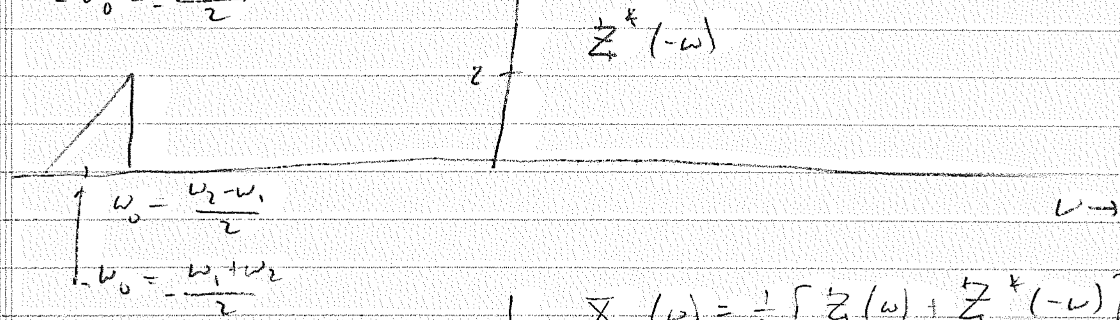
$$\updownarrow$$

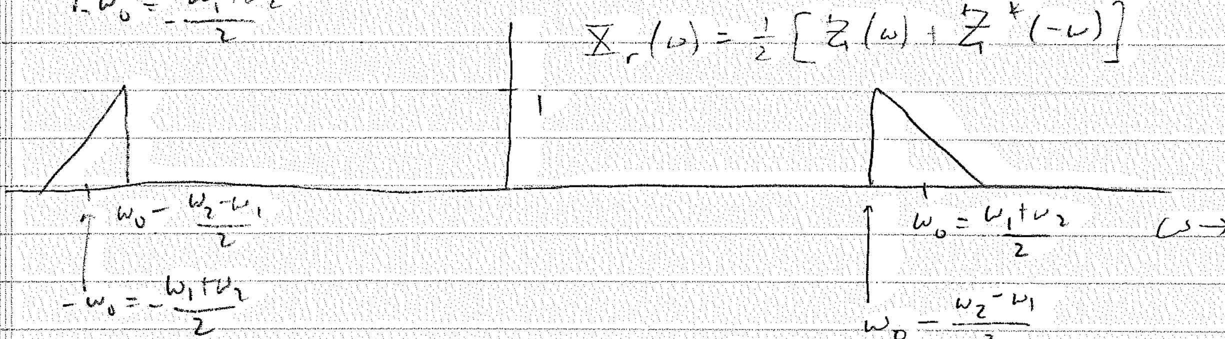
$$X_r(\omega) = \frac{1}{2}\left[ Z(\omega) + \mathcal{F}\{z^*(t)\} \right]$$

$$\mathcal{F}\{z^*(t)\} = \int_{-\infty}^{+\infty} z^*(t) e^{-j\omega t} dt = \left[ \int_{-\infty}^{+\infty} z(t) e^{-j(-\omega)t} dt \right]^* = Z^*(-\omega)$$

$$\Longrightarrow \quad X_r(\omega) = \frac{1}{2}\left[ Z(\omega) + Z^*(-\omega) \right]$$




So $X_r(\omega) = X_c(\omega) \longleftrightarrow x_r(t) = x_c(t)$!

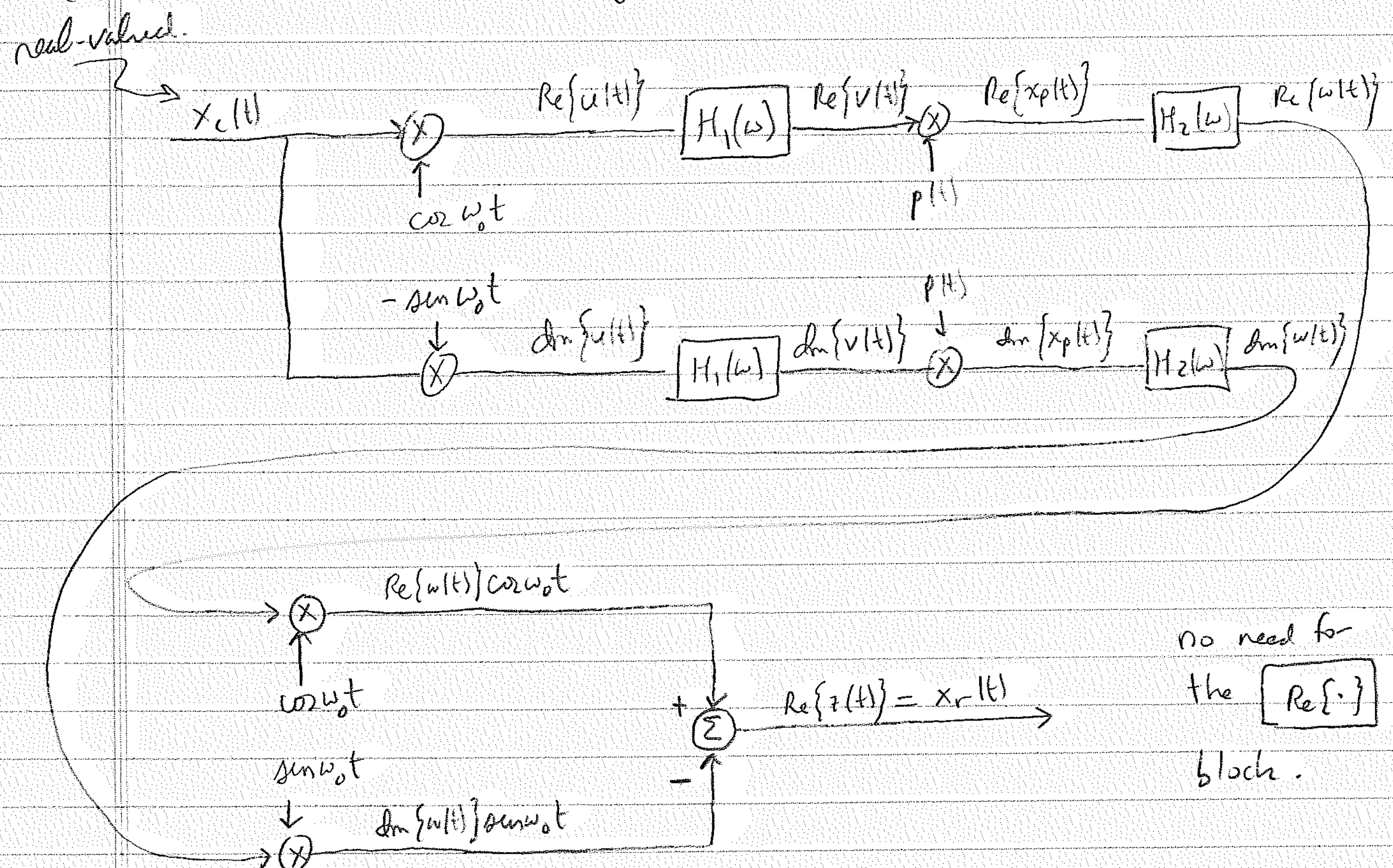
Notice that the sampling rate is <u>low</u>: the sampling frequency $= \omega_s = 2(\omega_2 - \omega_1)$

This might be vastly smaller than twice the highest frequency present in $x_c(t)$ which is $2\omega_2$.

We could even reduce the sampling rate to a sampling frequency of $\omega_s = \omega_2 - \omega_1$.

Then the pulses in $X_p(\omega)$ would be exactly adjacent to one another.

(b) $e^{j\phi} = \cos\phi + j\sin\phi$




no need for the $\boxed{Re\{\cdot\}}$ block.

$$z(t) = w(t)e^{j\omega_0 t} \implies Re\{z(t)\} = Re\{w(t)\}\cos\omega_0 t - Im\{w(t)\}\sin\omega_0 t$$