ECE 2200 and ENGRD 2220
Signals and Systems
Spring 2016
Final Exam
Wednesday May 18, 2016
7:00–9:30 PM
Olin Hall Room 165
No calculator!
Only the provided formula sheet!
Work alone!

1. (50 = 10 + 15 + 15 + 10 pts.) The goal of this problem is to compute the continuous-time Fourier transform of the unit step function and then use the result. Several methods to compute the transform are proposed. You must try each method and be clear about where the method fails. One of the methods will be successful with straightforward calculations.

   The unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}. \tag{148}$$

   There are various definitions of $u(t = 0)$ but changing the value of a continuous-time signal at one time point does not change any engineering result. The continuous time Fourier transform $U(\omega)$ of $u(t)$ is

$$U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \tag{149}$$

   (a) The first method is to use the analysis equation of the Fourier transform directly:

$$U(\omega) = \int_{-\infty}^{+\infty} u(t)\exp(-j\omega t)\mathrm{d}t. \tag{150}$$

   Is this a successful approach to computing $U(\omega)$?

   (b) The second method uses Fourier transform properties. Define $p_T(t) \leftrightarrow P_T(\omega)$ by

$$p_T(t) = \begin{cases} 1, & |t| < T \\ 0, & \text{otherwise} \end{cases}. \tag{151}$$

   Please plot $p_T(t)$. Then

$$u(t) = \lim_{T\to\infty} p_T(t - T). \tag{152}$$

   Is this a successful approach to computing $U(\omega)$?

   (c) The third method also uses Fourier transform properties. Define $g_\gamma(t) \leftrightarrow G_\gamma(\omega)$ by

$$g_\gamma(t) = \begin{cases} -\exp(\gamma t), & t < 0 \\ \exp(-\gamma t), & t \geq 0 \end{cases} \tag{153}$$

   where $\gamma > 0$. Please notice that there are two minus signs. Please plot $g_\gamma(t)$ for $\gamma = 1$ and for $\gamma = 0.1$. Then

$$u(t) = \lim_{\gamma\to 0} \frac{1}{2}\left(1 + g_\gamma(t)\right). \tag{154}$$

   Is this a successful approach to computing $U(\omega)$?

(d) Suppose that $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$ are the input and output of a system, respectively, and

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)\mathrm{d}\tau. \tag{155}$$

Please derive (do not just state) a formula for $Y(\omega)$ in terms of $X(\omega)$. Please simplify your result! Hint: Rewrite Eq. 155 as a convolution with an impulse $h(t) \leftrightarrow H(\omega)$. Then $Y(\omega) = H(\omega)X(\omega)$.

2. ($30 = 20 + 10$ pts.) The Discrete Fourier Transform (DFT) pair of synthesis and analysis equations is

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi}{N}nk\right) \tag{156}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \exp\left(j\frac{2\pi}{N}nk\right). \tag{157}$$

Please note that Eqs. 156 and 157 imply that $X[k+N] = X[k]$ and $x[n+N] = x[n]$ so both are periodic.

(a) Please compute the DTF of $x[n]$ defined on the interval $n \in \{0, 1, \ldots, N-1\}$ by

$$x[n] = \exp(-\alpha n) \tag{158}$$

where α is a real number and $\alpha > 0$.

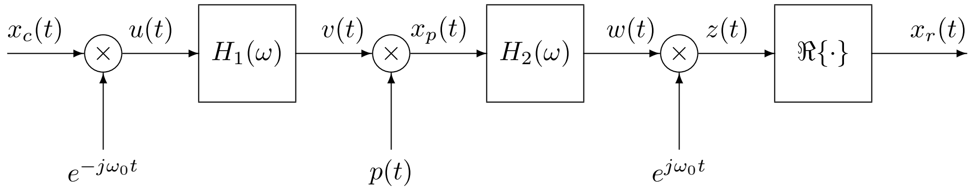
(b) Denote the answer to Problem 2a by $X[k]$. Define $y[n] \leftrightarrow Y[k]$ ($Y[k]$ is the DFT of $y[n]$) by

$$Y[k] = \exp\left(-j\frac{2\pi}{N}n_0 k\right) X[k] \tag{159}$$

for $n_0 = 3$. Assume that $N-1 > n_0$. Assume that α is roughly $1/(N-1)$. Please give an accurate plot of $y[n]$ for $n \in \{0, \ldots, N-1\}$.

3. ($70 = 12 \times 4 + 22$ pts.) This problem considers the situation where you need to sample a bandpass signal, i.e., a signal for which the Fourier transform is non zero only in a limited range of frequencies that does not include 0. You can always use the Nyquist sampling rate, i.e., twice the highest frequency in the signal, but that seems wasteful if the Fourier transform is non zero over only a small range of frequencies but the highest frequency is very large. The approach taken here is a combination of a DSB-SC communication system and a sampling system. However, instead of multiplying by $\sqrt{2}\cos(2\pi f_c t)$, the block diagram includes multiplication by $\exp(-j\omega_0 t)$ and $\exp(j\omega_0 t)$. Because of Euler's formula, this means that both multiplication by sin and by cos is included. Note that using $\exp(\pm j\omega_0 t)$ implies that all of the intermediate signals are complex valued.

Consider the following block diagram:



where

(a) $x_c(t)$ is real.

(b) $X_c(\omega) = 0$ except for $\omega_1 \le |\omega| \le \omega_2$.

(c) $\omega_0 = (\omega_1 + \omega_2)/2$.

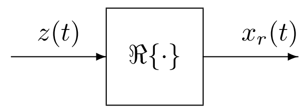
(d) $H_1(\omega) = 1$ if $|\omega| \le (\omega_2 - \omega_1)/2$ and $H_1(\omega) = 0$ otherwise.

(e) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\frac{2\pi}{\omega_s})$.

(f) $\omega_s = 2(\omega_2 - \omega_1)$. $T_s = 2\pi/\omega_s$.

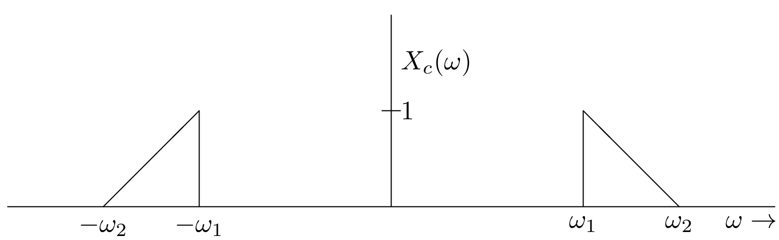
(g) $H_2(\omega) = 2T_s$ if $|\omega| \le (\omega_2 - \omega_1)/2$ and $H_2(\omega) = 0$ otherwise.

(h)



means that $x_r(t) = \Re\{z(t)\}$ which implies that $X_r(\omega) = \frac{1}{2}[Z(\omega) + Z^*(-\omega)]$. This is an important hint!

Let $X_c(\omega)$ be as shown in the following graph:

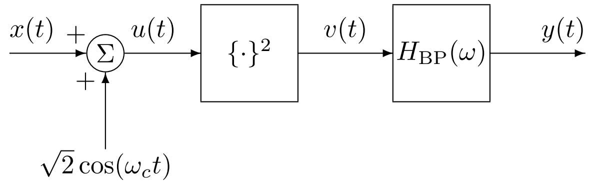


(a) Plot the Fourier transforms of all the signals and the frequency responses of all the filters on the block diagram: $x_c(t)$, $\exp(-j\omega_0 t)$, $u(t)$, $H_1(\omega)$, $v(t)$, $p(t)$, $x_p(t)$, $H_2(\omega)$, $w(t)$, $\exp(j\omega_0 t)$, $z(t)$, and $x_r(t)$. Does this sampling and reconstruction system make $x_r(t)$ equal to $x_c(t)$? Could the sampling rate ω_s be reduced and still have a successful system?

(b) Please assume that $x_c(t)$ is a real-valued signal. Please give an equivalent block diagram that uses $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ in place of $\exp(\pm j\omega_0 t)$ and every signal is real valued.

4. (50 = 12 + 12 + 12 + 14 pts.) This problem concerns building modulators and demodulators for Double-Side-Band Suppressed-Carrier (DSB-SC) communication systems using squaring operations.

   (a) Let $x(t) = A_c m(t)$ where $m(t)$ is bandlimited to ω_m, i.e., $|M(\omega)| = 0$ for $|\omega| > \omega_m$. Consider the following block diagram:

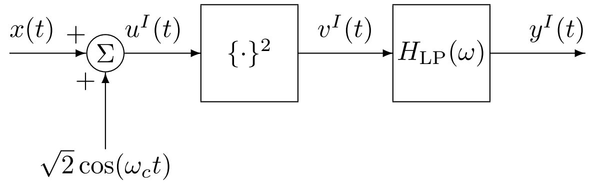
   

   where $\omega_c \gg \omega_m$ and

   $$H_{\mathrm{BP}}(\omega) = \begin{cases} 1, & |\omega - \omega_c| \leq \omega_m \\ 1, & |\omega + \omega_c| \leq \omega_m \\ 0, & \text{otherwise} \end{cases} .$$

   The block labeled "$\{\cdot\}^2$" has output $v(t)$ that is the square of the input $u(t)$: $v(t) = u^2(t)$. This block diagram is attractive because it is easier to build a squarer than to build a multiplier. Give a formula for $y(t)$.

   (b) Based on the successful use of a squaring operation in the modulator of Problem 4a, we decide to use the same squaring operation in the demodulator. Let $x(t) = A_c m(t)\sqrt{2}\cos(\omega_c t)$ where $m(t)$ is bandlimited to ω_m, i.e., $|M(\omega)| = 0$ for $|\omega| > \omega_m$, and $\omega_c \gg \omega_m$. Consider the following block diagram:

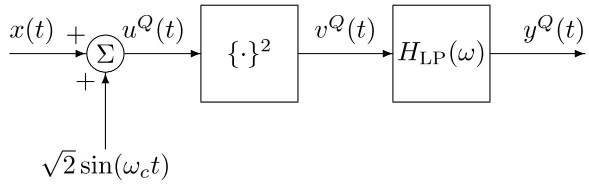
   

   where $\omega_c \gg \omega_m$ and

   $$H_{\mathrm{LP}}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_m \\ 0, & \text{otherwise} \end{cases} .$$

   Give a formula for $y^I(t)$. Does this block diagram work as a demodulator, that is, is $y^I(t)$ proportional to $m(t)$?

   (c) Due to the failure in Problem 4b, we have to think hard and it seems natural to consider also the block diagram with cos replaced by sin. Let $x(t) = A_c m(t)\sqrt{2}\cos(\omega_c t)$ where $m(t)$ is bandlimited to ω_m, i.e., $|M(\omega)| = 0$ for $|\omega| > \omega_m$, and $\omega_c \gg \omega_m$. Consider the following block diagram:

   where $\omega_c \gg \omega_m$ and

   $$H_{\mathrm{LP}}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_m \\ 0, & \text{otherwise} \end{cases} .$$

   Give a formula for $y^Q(t)$.

   (d) Combining the results of Problems 4b and 4c, draw a block diagram of a successful DSB-SC demodulator using squaring operations instead of multipliers.