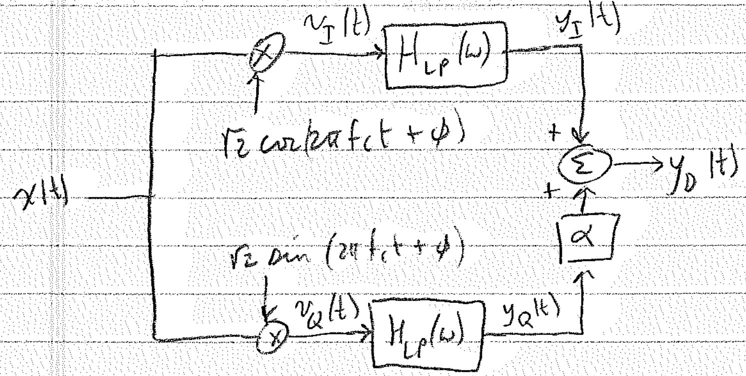
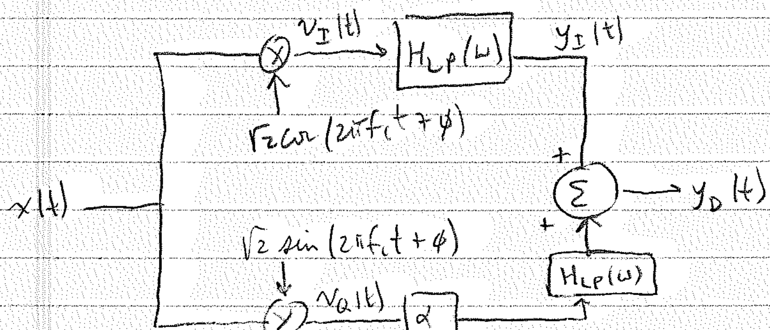
ECE 2200 / Spring 2016 / Preliminary Exam 2 / Solution

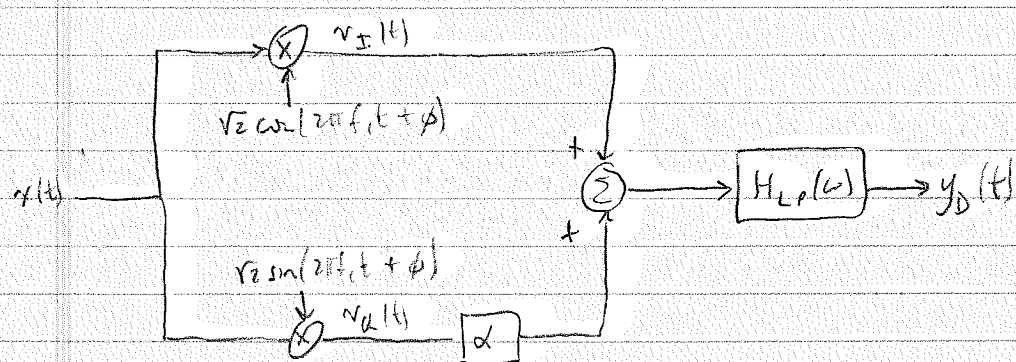
p070.EE440p017



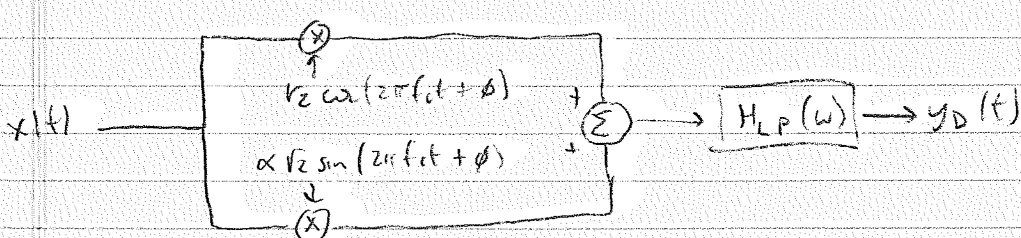
is equivalent to the following by commutative property



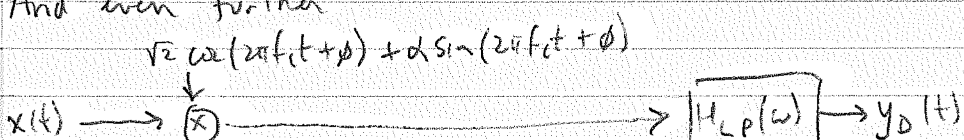
is equivalent to the following by distributive property

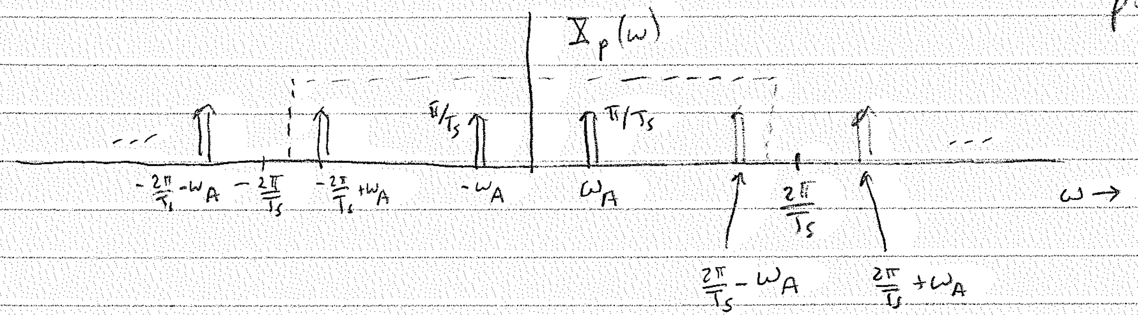


Could simplify further:



And even further





$$\omega_B = \frac{2\pi}{T_s} - \omega_A \Rightarrow \frac{2\pi}{T_s} = \omega_A + \omega_B \Rightarrow \frac{T_s}{2\pi} = \frac{1}{\omega_A+\omega_B} \Rightarrow \boxed{T_s = \frac{2\pi}{\omega_A+\omega_B}}$$

$$\frac{2\pi}{T_s} - \omega_A < \omega_H < \frac{2\pi}{T_s} + \omega_A \Rightarrow \omega_A + \omega_B - \omega_A < \omega_H < \omega_A + \omega_B + \omega_A$$

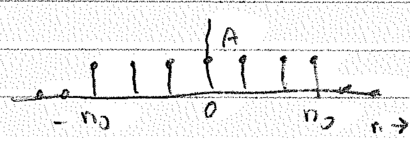
$$\Rightarrow \boxed{\omega_B < \omega_H < 2\omega_A + \omega_B}$$

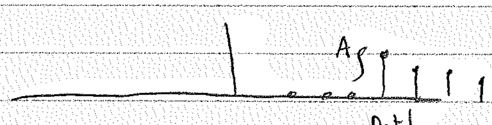
$$x[n] = \begin{cases} A & |n| \le n_0 \\ A\rho^{|n|-n_0} & |n| > n_0 \end{cases} \qquad A \in \mathbb{R},\ \rho \in \mathbb{R},\ |\rho| < 1$$

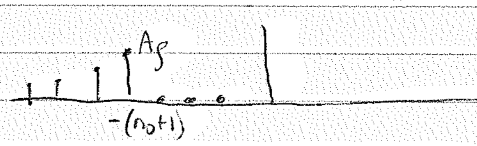
$$y[n] = \begin{cases} 1 & |n| \le n_0 \\ 0 & |n| > n_0 \end{cases} \longleftrightarrow Y(\Omega) = \frac{\sin(\Omega(2n_0+1)/2)}{\sin(\Omega/2)}$$

$$z[n] = \rho^n u[n] \longleftrightarrow Z(\Omega) = \frac{1}{1-\rho e^{-j\Omega}}$$

(a) $$x[n] = \underset{①}{A y[n]} + \underset{②}{A\rho^{n-n_0} \underbrace{u[n-n_0-1]}_{\substack{=1 \text{ for } n \ge n_0+1 \\ =0 \text{ otherwise}}}} + \underset{③}{A\rho^{-n-n_0} \underbrace{u[-n-n_0-1]}_{\substack{=1 \text{ for } n \le -n_0-1 \\ =0 \text{ otherwise}}}}$$

① = 
$d[n] = Ay[n] \leftrightarrow D(\Omega) = A\,Y(\Omega)$

② 

③ 

$x[n] \leftrightarrow X(\Omega)$

$y[n] = x[n-n_0]$

$$\Rightarrow Y(\Omega) = \sum_{n=-\infty}^{+\infty} y[n] e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x[n-n_0] e^{-j\Omega n} \qquad m = n-n_0$$

$$= \sum_{m=-\infty}^{+\infty} x[m] e^{-j\Omega(m+n_0)} = e^{-j\Omega n_0} X(\Omega)$$

② : $b[n] = A\rho^{n-n_0} u[n-n_0-1]$

$$c[n] = b[n+(n_0+1)] = A\rho^{n+(n_0+1)-n_0} u[n+(n_0+1)-n_0-1] = A\rho^{n+1} u[n]$$

$$= A\rho\, \rho^n u[n] = A\rho\, z[n] \longleftrightarrow C(\Omega) = A\rho\, Z(\Omega)$$

Also, $C(\Omega) = e^{j\Omega(n_0+1)} B(\Omega)$

$$\Longrightarrow B(\Omega) = e^{-j\Omega(n_0+1)} C(\Omega) = e^{-j\Omega(n_0+1)} A\rho\, Z(\Omega)$$

(3): $f[n] = b[-n]$

$$F(\Omega) = \sum_{n=-\infty}^{+\infty} f[n] e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} b[-n] e^{-j\Omega n} \qquad m = -n$$

$$= \sum_{m=-\infty}^{+\infty} b[m] e^{-j\Omega(-m)} = \sum_{m=-\infty}^{+\infty} b[m] e^{-j(-\Omega)m} = B(-\Omega)$$

$$x[n] = d[n] + b[n] + f[n]$$
$$\updownarrow$$
$$X(\Omega) = D(\Omega) + B(\Omega) + F(\Omega)$$

$$= A\,Y(\Omega) + B(\Omega) + B(-\Omega)$$

$b[n] \in \mathbb{R}$

$$B(\Omega) = \sum_{n=-\infty}^{+\infty} b[n] e^{-j\Omega n}$$

$$\Rightarrow B^*(\Omega) = \sum_{n=-\infty}^{+\infty} b^*[n] e^{+j\Omega n} = \sum_{n=-\infty}^{+\infty} b[n] e^{-j(-\Omega)n} = B(-\Omega)$$

Alternatively, look at the formulas for $B(\Omega)$ and $Z(\Omega)$ and see that $B^*(\Omega) = B(-\Omega)$.

Either way,

$$X(\Omega) = A\,Y(\Omega) + B(\Omega) + B^*(\Omega)$$

$$= A\,Y(\Omega) + 2\,\mathrm{Re}\{B(\Omega)\}$$

$$= A\,Y(\Omega) + 2\,\mathrm{Re}\left\{\exp(-j\Omega(n_0+1))\,A_\rho\,Z(\Omega)\right\}$$

$$= A\,Y(\Omega) + 2A_\rho\,\mathrm{Re}\left\{\exp(-j\Omega(n_0+1))\,\frac{1}{1-\rho e^{-j\Omega}}\right\}$$

(b) In the calculations of (a) it is shown that

$$x[n] \in \mathbb{R} \implies X^*(\Omega) = X(-\Omega).$$

$$x[n] = x[-n] \quad (\text{i.e., } x[n] \text{ is even}) \implies X(\Omega) = X(-\Omega).$$

Therefore,

$$X^*(\Omega) = X(-\Omega) = X(\Omega).$$

Therefore

$$X(\Omega) \in \mathbb{R}.$$

(c) So it is necessary to show that $X(\Omega)$ from (a) is real.

This is straightforward since

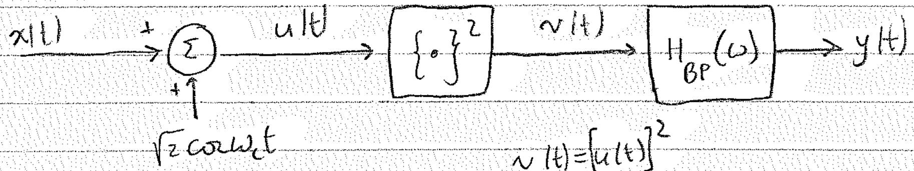
$$X(\Omega) = \underset{\text{real}}{A}\ \underset{\text{real}}{Y(\Omega)} + \underset{\text{real}}{2A}\underset{\text{real}}{\rho}\ \underset{\text{real}}{\mathrm{Re}}\left\{\exp(-j\Omega(n_0+1))\frac{1}{1-\rho e^{-j\Omega}}\right\}$$

So $X(\Omega) \in \mathbb{R}$

$$\cos^2\alpha = \tfrac{1}{2}(1+\cos 2\alpha) \qquad \cos\alpha \sin\beta = \tfrac{1}{2}\sin(\alpha+\beta) - \tfrac{1}{2}\sin(\alpha-\beta)$$

$$\sin^2\alpha = \tfrac{1}{2}(1-\cos 2\alpha)$$

a)

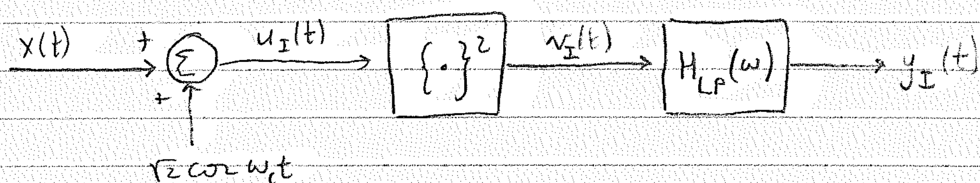


$v(t) = [u(t)]^2$

$$x(t) = A_c m(t)$$
$$u(t) = A_c m(t) + \sqrt{2}\cos\omega_c t$$
$$v(t) = A_c^2 m^2(t) + 2A_c m(t)\sqrt{2}\cos\omega_c t + 2\cos^2\omega_c t$$
$$= A_c^2 m^2(t) + 2A_c m(t)\sqrt{2}\cos\omega_c t + 1 + \cos 2\omega_c t$$
$$= \underbrace{1 + A_c^2 m^2(t)}_{\text{below filter's cutoff}} + \underbrace{2A_c m(t)\sqrt{2}\cos\omega_c t}_{\text{in filter's passband}} + \underbrace{\cos 2\omega_c t}_{\text{above filter's cutoff}}$$

$$y(t) = 2A_c m(t)\sqrt{2}\cos\omega_c t$$

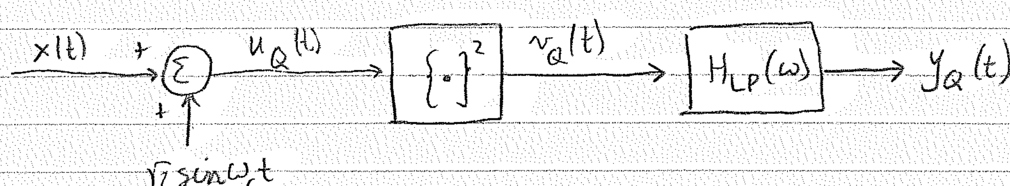
b)



$$x(t) = A_c m(t)\sqrt{2}\cos\omega_c t$$
$$u_I(t) = (A_c m(t) + 1)\sqrt{2}\cos\omega_c t$$
$$v_I(t) = (A_c^2 m^2(t) + 2A_c m(t) + 1)\, 2\cos^2\omega_c t$$
$$= (A_c^2 m^2(t) + 2A_c m(t) + 1)(1 + \cos 2\omega_c t)$$
$$= \underbrace{(A_c^2 m^2(t) + 2A_c m(t) + 1)}_{\text{in filter's passband}} + \underbrace{(A_c^2 m^2(t) + 2A_c m(t) + 1)\cos(2\omega_c t)}_{\text{in filter's stop band}}$$

$$y_I(t) = \underbrace{A_c^2 m^2(t) + 1}_{\text{junk!}} + \underbrace{2A_c m(t)}_{\text{desired signal}} = [A_c m(t) + 1]^2$$

c)



$$x(t) = A_c m(t)\sqrt{2}\cos\omega_c t$$
$$u_Q(t) = A_c m(t)\sqrt{2}\cos\omega_c t + \sqrt{2}\sin\omega_c t$$

$$v_Q(t) = A_c^2 m^2(t)\, 2\cos^2\omega_c t + 2A_c m(t)\sqrt{2}\cos\omega_c t\,\sqrt{2}\sin\omega_c t + 2\sin^2\omega_c t$$

$$= A_c^2 m^2(t)\,(1+\cos 2\omega_c t) + 4A_c m(t)\left[\tfrac{1}{2}\sin(2\omega_c t) - \tfrac{1}{2}\underbrace{\sin 0}_{=0}\right] + \left[1 - \cos 2\omega_c t\right]$$

$$= A_c^2 m^2(t) + 1 + A_c^2 m^2(t)\cos 2\omega_c t + 2A_c m(t)\sin 2\omega_c t - \cos 2\omega_c t$$

$$= \underbrace{A_c^2 m^2(t) + 1}_{\text{in filter's passband}} + \underbrace{\left(A_c^2 m^2(t) - 1\right)\cos 2\omega_c t + 2A_c m(t)\sin 2\omega_c t}_{\text{in filter's stopband}}$$

$$y_Q(t) = A_c^2 m^2(t) + 1$$

(d) Note that $y(t)$ defined by $y(t) = y_I(t) - y_Q(t)$ has the value

$$y(t) = y_I(t) - y_Q(t)$$

$$= A_c^2 m^2(t) + 1 + 2A_c m(t) - \left[A_c^2 m^2(t) + 1\right]$$

$$= 2A_c m(t)$$

which is the signal that we want.

The corresponding block diagram is