10 Sept. 2014 - Due Wed., 17 Sept. 2014 in Dropbox
20% grade reduction per late day submission
There are 9 problems to this homework

1. OK!

2.

(a) Generalizing the calculation that we got in class,

$$P(\text{Row k selected}|\text{Seat 15 chosen}) = \frac{P(\text{Seat 15 chosen}|\text{Row k selected}) \cdot P(\text{Row k selected})}{P(\text{Seat 15 chosen})}$$

The first probability in the numerator is $1/(k+10)$, where k is the row number. The second probability in the numerator is simply $1/30$. The probability in the denominator is the total probability $P(\text{Seat 15 is chosen})$ and is given by (like we saw in class):

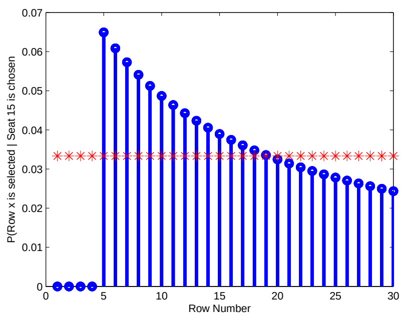
$$P(\text{Seat 15 is chosen}) = \sum_{i=5}^{30} \frac{1}{i+10}\frac{1}{30}.$$

The following MATLAB code can be used:

```
probSeat15Chosen = 0;
for i = 5 : 30
    probSeat15Chosen = probSeat15Chosen + 1/(i+10) * (1/30);
end
probRow_k_GivenSeat15(1:4) = 0; % 0 Prob for first 4 rown: no seat 15!
for rowNumber = 5 : 30
    probRow_k_GivenSeat15(rowNumber) = (1/(rowNumber + 10)) * (1/30) / ...
        probSeat15Chosen;
end
stem(probRow_k_GivenSeat15, 'linewidth', 3)
xlabel('Row Number')
ylabel('P(Row x is selected | Seat 15 is chosen')
hold on
% Identify the a priori probability P(Row k is selected)
probRow20Selected(1:30) = 1/30;
plot(probRow20Selected, '*r', 'MarkerSize', 10)
%
```

And the plot gives you the image shown on the next page, where the red asterisks identify the *a priori* probability that row i is chosen (which is 1/30)

Clearly, the action of first choosing Seat 15 affects the probabilities of the events of choosing Row i. Do you see why? There are fewer seats to choose from in the lower numbered rows than in the higher numbered rows: so if we first choose Seat 15, it is more likely that it belonged to a lower row, and so it is more likely that a lower row number will then be selected.



(b) (i) To Solve.
(ii) To Solve.

3. Since the events $A \cap B^c$ and $A^c \cap B$ are disjoint, we have using the additivity axiom repeatedly,

$$P\Big((A \cap B^c) \cup (A^C \cap C)\Big) = P(A \cap B^c) + P(A^c \cap B) = P(A) - P(A \cap B) + P(B) - P(A \cap B).$$

4. Let D_i be the i^{th} diode that is chosen that is defective. Then,

$$P(D_1 \cap D_2) = 1 - P(D_1^c \cap D_2^c) = 1 - P(D_1^c)P(D_2^c|D_1^c),$$

and we also have that $P(D_1^c) = 43/50$ and $P(D_2^c|D_1^c) = 42/49 = 6/7$. This is because after the first diode is selected and it is found NOT to be defective, there are 49 diodes remaining, of which 7 are defective and 42 are not. Then,

$$P(D_1 \cap D_2) = 1 - \frac{43}{50} \cdot \frac{6}{7} = \frac{46}{175} \approx 0.26.$$

Or a quick way to solve for this is the following:

$$P(\text{at least one is defective}) = 1 - P(\text{both are good})$$

and this is simply:

$$1 - \frac{43}{50}\frac{42}{49}.$$

5.

(a)

$$P(1^{st} \text{ is red } \cap 2^{nd} \text{ is blue}) = P(1^{st} \text{ is red}) \cdot P(2^{nd} \text{ is blue} \mid 1^{st} \text{ is red}),$$
$$\text{with } P(1^{st} \text{ is red}) = \frac{3}{12},$$
$$P(2^{nd} \text{ is blue} \mid 1^{st} \text{ is red}) = \frac{5}{11},$$
$$\text{and so the final result is } \frac{5}{44}.$$

(b)

$$P(\text{2nd} = \text{white}) = \frac{4}{12} = \frac{1}{3}.$$

(c) Same as part (b).

6.

(a) For some even integer n, there will be $n/2$ 0's and $n/2$ 1's. The number of ways that the 0's and 1's can be arranged is

$$\frac{n!}{(\frac{n}{2})!(\frac{n}{2})!}$$

(b) We want to find the smallest n such that the expression in part (a) above is ≥ 100. If $n = 8$, there are 70 different characters, and if $n = 10$, there are 252 different characters. Therefore, the minimum length of the codeword would need to be 10 bits.

7.

$$P(\text{PC is defective}) = P(\text{PC is defective} \mid A) \cdot P(A) + P(\text{PC is defective} \mid B) \cdot P(B)$$
$$= 0.15 \cdot \frac{1}{1.15} + 0.05 \cdot \frac{0.15}{1.15} = 0.137.$$

8.

(a) P(0 received) = P(0 transmitted) $\cdot$ P(0 received | 0 transmitted) + P(1 transmitted)$\cdot$ P(0 received | 1 transmitted) = $\frac{1}{2}0.95 + \frac{1}{2}0.10 = 0.525$.

Now, P(1 received) = P(0 transmitted) $\cdot$ P(1 received | 0 transmitted) + P(1 transmitted) $\cdot$ P(1 received | 1 transmitted) = $\frac{1}{2}0.05 + \frac{1}{2}0.90 = 0.475$.

In passing, note that P(1 received) = 1 - P(0 received)= 0.475.

(b) P(1 transmitted | 0 received)=

$$= \frac{\text{P(1 transmitted} \cap \text{0 received)}}{\text{P(0 received)}}$$

$$= \frac{\text{P(1 transmitted)}\cdot\text{P(0 received | 1 transmittted)}}{\text{P(0 received)}}$$

$$= \frac{0.5 * 0.1}{0.525} \approx 0.095$$

Using a similar reasoning, we get P(0 transmitted | 1 received)

$$= \frac{0.5 \cdot 0.05}{0.475} \approx 0.0526$$

(c) The P(error) is given by

$$P(\text{1 received}|\text{0 sent}) \cdot P(\text{0 sent}) + P(\text{0 received}|\text{1 sent}) \cdot P(\text{1 sent}) = .075.$$

9.

(a) We use the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B|A)}{P(B)}.$$

Since all crows are black, we have $P(B) = 1 - q$. Also, $P(A) = p$, and $P(B|A) = 1 - q = P(B)$, since the probability of observing a (black) crow is not affected by the truth of our hypothesis. We conclude that $P(A|B) = P(A) = p$. This, the new evidence, while compatible with the hypothesis "all crows are white," does not change our beliefs about its truth.

(b) Here also,

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)P(C|A)}{P(C)}.$$

Given the event A, a cow is observed with probability q, and it must be white. This means that $P(C|A) = q$. Given the event A^c, a cow is observed with probability q, and it is white with probability 1/2. Thus, $P(C|A^c) = q/2$. Using the total probability theorem,

$$P(C) = P(A)P(C|A) + P(A^c)P(C|A^c) = pq + (1-p)\frac{q}{2}.$$

Hence,

$$P(A|C) = \frac{pq}{pq + (1-p)\frac{q}{2}} = \frac{2p}{1+p} > p.$$

This means that the observation of a white crow makes the hypothesis "all cows are white" more likely to be true.