10 Sept. 2014 - Due Wed., 17 Sept. 2014 in Dropbox
20% grade reduction per late day submission
There are 9 problems to this homework

1. Read the book, pages 1-87.

2. Let's revisit the example that was given in class about the auditorium seats, where the row was first selected randomly, then the seat was selected randomly.

(a) As discussed in class, it seemed counter intuitive that, given that seat 15 was chosen, this reduced the probability that row 20 was selected. After all, a 15^{th} seat only exists in rows 5-30, so there are fewer rows to choose from, and so there should be an increase in the probability of choosing row 20 knowing that seat 15 was chosen. To see this, calculate (using MATLAB or other coding language) the probabilities that rows 1-30 were chosen given that seat 15 was chosen, and plot the results as a function of row number (*i.e.*, you will calculate 30 probabilities like we did in class - well one less because we did this calculation for one case, and 4 of those probabilities are trivial). You should observe in the graph that the event that seat 15 was chosen makes some rows much more probable than others. Comment on the results (explain what you see, not just something like "the curve is high then low"). Submit the plot and code with your homework.

(b) Let's make the problem a little more complicated. Instead of first selecting the row randomly followed by selecting a seat randomly, each ticket for the draw (one ticket per seat) has written on it the row number and the seat number. All of the tickets are put in a box and one is chosen randomly to determine the winner. So, for example, a randomly picked ticket would read (Row 6, Seat 5).

(i) The box contains one ticket for all of the seats in the auditorium. A ticket is chosen randomly. You are told that the randomly chosen ticket has "Row 20" written on it. Calculate the probability that the ticket also has "Seat 15" written on it.

(ii) The ticket is put back in the box so that the box contains once again one ticket for all seats in the auditorium. A ticket is chosen randomly. You are told that the randomly chosen ticket has "Seat 15" written on it. Calculate the probability that the ticket also has "Row 20" written on it.

3. *Problem 1.10.* Show the formula:

$$P\left((A \cap B^c) \cup (A^c \cap B)\right) = P(A) + P(B) - 2P(A \cap B)$$

which gives the probability that exactly one of the events A and B will occur. [Compare with the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, which gives the probability that at least one of the events A and B will occur.]

4. A box contains 50 diodes. You know that there are 7 defective diodes in that box. If you select two diodes at random without replacement (you pick one diode, you don't put it back in the box, and you pick another diode), what is the probability that at least one of those is defective?

5. An urn contains 3 red balls, 4 white balls, 5 blue balls. You select two balls sequentially, without replacement.

(a) What is the probability that the first ball is red and the second ball is blue?

(b) What is the probability of selecting a white ball on the second draw if the first ball is replaced before the second ball is drawn?

(c) What is the probability of selecting a white ball on the second draw if the first ball is not replaced before the second ball is drawn?

6. In a certain communication system, text messages are transmitted by representing each character with an n-bit binary codeword. Suppose that this communication system requires that there are always an equal number of 0's and 1's that is transmitted. To accomplish this, the communication system uses a codebook that consists only of those n-bit words that have exactly $n/2$ 0's and $n/2$ 1's (where n is an integer). For example, for $n = 4$, there are exactly 6 four-bit codewords that have exactly two 0's and two 1's that make up the codebook: $(1100), (1010), (1001), (0110), (0101), (0011)$. This means that for $n = 4$, , *i.e.*, with four bit codewords, we can represent an alphabet of only six characters.

(a) Write an expression for the number of codewords that have 1/2 0's and 1/2 1's for some arbitrary even integer n.

(b) What is the minimum length of codeword that we would be if the codebook needs to represent at least 100 different characters?

7. *Bayes' Theorem.* Some company manufactures computers at two different locations in the world (location A and location B), and those computers are shipped to various retail outlets. Let's consider only one such outlet, Mike's Computers. You know that 15% of the computers that are manufactured at location A are shipped to Mike's Computers in a defective condition, and 5% of the computers that are manufactured at location B are shipped defective to Mike's Computers. If location A manufactures 1 million PC's per year, and location B manufactures 150,000 PC's per year, what is the probability of buying a defective PC from Mike's Computers?

8. *Total Probability / Bayes' Theorem.* A communication system sends binary data (0's or 1's). This data is detected at the receiver, and the receiver sometimes makes mistakes: it sometimes detects a 0 when a 1 is sent, and vice-versa. Suppose that this communication system has the following four probabilities:+

$$P(0 \text{ received} \mid 0 \text{ transmitted}) = 0.95$$
$$P(1 \text{ received} \mid 0 \text{ transmitted}) = 0.05$$
$$P(0 \text{ received} \mid 1 \text{ transmitted}) = 0.1$$
$$P(1 \text{ received} \mid 1 \text{ transmitted}) = 0.9$$

(a) Assume that 0's and 1's have equal probability of being transmitted (*i.e.*, $P(0 \text{ transmitted}) = P(1 \text{ transmitted}) = \frac{1}{2}$). Evaluate $P(0 \text{ received})$ and $P(1 \text{ received})$.

(b) Suppose that a 0 is detected at the receiver end. What is the probability that a 1 was transmitted? Also, if a 1 is detected at the receiver end, what is the probability that a 0 was transmitted?

(c) What is the probability that the detected bit is not equal to the transmitted bit? This result is the overall probability of error of the receiver.

9. *Problem 1.26.* Consider a statement whose truth is unknown. If we see many examples that are compatible with it, we are tempted to view the statement as more probable. Such reasoning is often referred to as inductive inference (in a philosophical, rather than mathematical sense).

Consider now the statement that "all cows are white." An equivalent statement is that "everything that is not white is not a cow." We then observe several black crows. Our observations are clearly compatible with the statement, but do they make the hypothesis "all cows are white" more likely? To analyze such a situation, we consider a probabilistic model. Let us assume that there are two possible states of the world, which we model as complementary events:

A: All cows are white,

A^c: 50% of all cows are white.

Let p be the *a priori* probability $P(A)$ that all cows are white. We make an observation of a cow or a crow, with probability q and $1 - q$, respectively, independent of whether event A occurs or not. Assume that $0 < p < 1$, $0 < q < 1$, and that all crows are black.

(a) Given the event $B =$ a black crow was observed, what is $P(A|B)$?

(b) Given the event $C =$ a white cow was observed, what is $P(A|C)$?