ECE 303 - Homework 12 Solutions

12.1

a) $R_{rad} = 73\ \Omega$ (for half-wave dipoles)

$$Re\{Z_A\} = R_{diss} + R_{rad} \Rightarrow R_{diss} = 27\ \Omega.$$

b) $$\eta_{rad} = \frac{R_{rad}}{R_{rad} + R_{diss}} = 0.73$$

c) $$G(\theta,\phi) = 1.64 \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\eta_{rad} \Rightarrow G_1(\theta_1 = 30°, \phi_1 = 0) = 0.286\ \eta_{rad}$$

d) $$A(\theta,\phi) = \frac{\lambda^2}{4\pi} G(\theta,\phi) \Rightarrow A_2(\theta_2 = 150°, \phi_2 = 0) = G_2(\theta_2 = 150°, \phi_2 = 0)\frac{\lambda^2}{4\pi}$$

$$= 0.286 \frac{\lambda^2}{4\pi}\eta_{rad}.$$

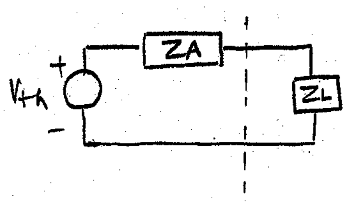
e) $$\frac{P_{out\text{-}2}}{P_{in\text{-}1}} = \frac{1}{4\pi L^2} G_1(\theta_1, \phi_1 = 0)\, A_2(\theta_2, \phi_2 = 0)$$

$$= \left(\frac{\lambda}{4\pi L}\right)^2 G_1(\theta_1 = 30°, \phi_1 = 0)\, G_2(\theta_2 = 150°, \phi_2 = 0) = \left(\frac{\lambda}{4\pi L}\right)^2 (0.286)\,\eta_{rad}^2$$

$$= 2.48 \times 10^{-11}$$

f) $$\frac{P_{out\text{-}1}}{P_{out\text{-}2}} = \frac{P_{out\text{-}2}}{P_{out\text{-}1}} \quad \{\text{reciprocity}\} = 2.48 \times 10^{-11}$$

g) One can make the following circuit for the receiving antenna:



Z_L is the impedance seen looking into the transmission line. In matched case $Z_L = Z_A^*$

and $P_{out\text{-}2} = \dfrac{|V_{th}|^2}{8\,Re\{Z_A\}}$. In the unmatched case:

$$P_{out\text{-}2} = \frac{1}{2} Re\left\{ \frac{V_{th}}{Z_A + Z_L} \cdot \frac{V_{th}^* \cdot Z_L^*}{(Z_A^* + Z_L^*)} \right\} = \frac{1}{2}\frac{|V_{th}|^2}{|Z_A + Z_L|^2} \cdot Re\{Z_L^*\}.$$

$$= \frac{1}{8}\frac{|V_{th}|^2}{Re\{Z_A\}} \cdot \left[\frac{4\,Re\{Z_A\}\,Re\{Z_L^*\}}{|Z_A + Z_L|^2}\right]$$

The factor in the square brackets equals 0.956

Therefore, compared to the matched case, the power received will be reduced by a factor of 0.956.

12.2.

a) $P = \frac{1}{2}\eta_0 |H|^2 \Rightarrow |H| = \sqrt{\frac{2P}{\eta_0}}$

b) $\int \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\iint \mu_0 \vec{H}\cdot d\vec{a} \Rightarrow V = -j\omega \iint \mu_0 \vec{H}\cdot d\vec{a} = -j\omega\mu_0 H \sin\theta (\pi a^2) N$

~~c) $P_{rec} = \frac{|V_{th}|^2}{8 R_{ad}}$ {For a matched load}. $V_{th} = V$~~

For a loop antenna $R_{ad} = \frac{\pi}{6}\eta_0 N^2 (ka)^4$

$$\Rightarrow P_{rec} = \frac{|V|^2}{8 R_{ad}} = \frac{\omega^2 \mu_0^2 |H|^2 \sin^2\theta \; \pi^2 a^4 N^2}{8 \frac{\pi}{6}\cdot \eta_0 \cdot N^2 \cdot (ka)^4}$$

d) P_{rec} can be written as $P_{rec} = P \cdot A(\theta,\phi)$

$$\Rightarrow A(\theta,\phi) = \frac{P_{rec}}{P} = \frac{\lambda^2}{4\pi}\cdot\frac{3}{2}\sin^2\theta$$

$$\Rightarrow A(\theta,\phi) = \frac{\lambda^2}{4\pi} G(\theta,\phi)$$

12.3.

a) We have $\oint \vec{E}\, d\vec{s} = -j\omega \iint \mu_0 \vec{H}\cdot d\vec{a}$

$$\Rightarrow I R_{diss} = -j\omega\mu_0 (-H_i \sin\theta)\pi a^2.$$

$$\Rightarrow I = \frac{j\omega\mu_0 H_i \sin\theta \; \pi a^2}{R_{diss}}$$

b) $$\vec{E}_{s\text{-}ff}(\vec{r}) = \hat{\phi}\, \frac{\eta_0 k^2 I (\pi a^2)}{4\pi r} \sin\theta\, e^{-jkr}$$ {where I is given in part (a)}

c) $$P_s = \oiint \frac{|\vec{E}_{s\text{-}ff}|^2}{2\eta_0} r^2 \sin\theta\, d\theta\, d\phi = \frac{\pi\eta_0}{12}(ka)^4 |I|^2$$

$$= \frac{\pi\eta_0}{12}(ka)^4 \frac{\omega^2\mu_0^2 (\pi a^2)^2 \sin^2\theta}{R_{diss}^2} |H_i|^2$$

d) $\sigma_s = \dfrac{P_s}{\frac{1}{2}\eta_0 |H_i|^2} = \dfrac{\pi}{6}(ka)^4 \dfrac{\omega^2 \mu_0^2 (\pi a^2)^2}{R_{diss}^2} \sin^2\theta$

12.4

a) From lecture #8, the induced dipole moment is

$\vec{P} = \hat{z}\, 4\pi \varepsilon_0 a^3 E_i$

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b) $\vec{E}_{s\text{-}ff}(\vec{r}) = \hat{\theta}\, j \dfrac{\eta_0 k (j\omega p)}{4\pi r} \sin\theta\, e^{-jkr} = -\hat{\theta}\, \dfrac{k^2 a^3}{r} E_i \sin\theta\, e^{-jkr}$

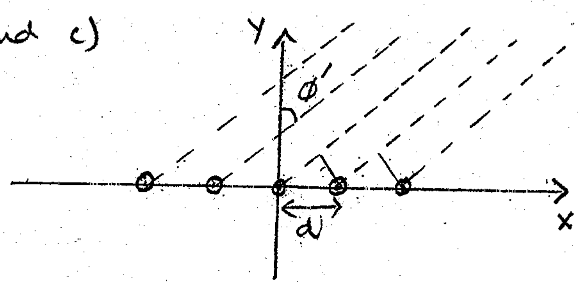
c) $P_s = \int_0^{2\pi}\int_0^{\pi} \dfrac{|\vec{E}_{s\text{-}ff}(\vec{r})|^2}{2\eta_0} r^2 \sin(\theta)\, d\theta\, d\phi = \dfrac{4\pi}{3\eta_0} k^4 a^6 |E_i|^2$

d) $\sigma_s = \dfrac{P_s}{\frac{1}{2\eta_0}|E_i|^2} = \dfrac{8}{3}(ka)^4(\pi a^2)$. $\Rightarrow$ since $ka \ll 1$, $\sigma_s \ll \pi a^2$

12.5

a) In the direction $\phi' = 0$ waves from all the dipoles are in phase and add constructively so there is a maxima in the $\phi' = 0$ direction irrespective of d and λ. as long as all dipoles have the same phase of the driving current.

b) and c)



In the ϕ' – direction, if the path difference between the waves emitted from adjacent dipoles is a multiple of the wavelength. λ then the waves from adjacent dipoles will add in phase.

$\Rightarrow$ for a maxima the condition is : $d \sin\phi' = n\lambda$ $\{ n = 0, 1, 2, \dots$

$\Rightarrow$ for the 1st maxima $n=1$ $\Rightarrow$ $d\sin\phi' = \lambda$ $\{$ or $kd\sin\phi' = 2\pi \}$.

$\Rightarrow$ for the 2nd maxima $n=2$ $\Rightarrow$ $d\sin\phi' = 2\lambda$ $\{$ or $kd\sin\phi' = 2\pi(2). \}$

d)

$$F(\theta=\tfrac{\pi}{2},\phi) = \sum_{i=0}^{N-1} e^{jkid\cos\phi} = \frac{\left(1-e^{jkdN\cos\phi}\right)}{\left(1-e^{jkd\cos\phi}\right)}$$

$$= \frac{e^{jkd\frac{N}{2}\cos\phi}}{e^{j\frac{kd}{2}\cos\phi}} \; \frac{\sin\frac{N}{2}(kd\cos\phi)}{\sin\frac{1}{2}(kd\cos\phi)}.$$

$$= \frac{e^{jkd\frac{N}{2}\sin\phi'}}{e^{j\frac{kd}{2}\sin\phi'}} \; \frac{\sin\frac{N}{2}(kd\sin\phi')}{\sin\frac{1}{2}(kd\sin\phi')} \qquad \left\{ \sin\phi' = \cos\phi \right\}.$$

$|F(\theta=\frac{\pi}{2},\phi)|^2$ will have maxima when the denominator term goes to zero, i.e $\sin\frac{1}{2}(kd\sin\phi')=0 \Rightarrow \frac{1}{2}(kd\sin\phi') = n\pi$ $\{n=0,1,..$

or $kd\sin\phi' = 2\pi n$ or $d\sin\phi' = n\lambda$. which is the same condition as derived earlier.

e) The nulls happen when the numerator in $|F(\theta=\frac{\pi}{2},\phi)|^2$ goes to zero WHILE the denominator is not zero.

$$\left.\begin{aligned} &\Rightarrow \sin\frac{N}{2}(kd\sin\phi') = 0 \\ &\Rightarrow \frac{N}{2}kd\sin\phi' = m\pi \end{aligned}\right\} \text{AND} \left\{\begin{aligned} &\sin\tfrac{1}{2}(kd\sin\phi') \neq 0 \\ &\Rightarrow kd\sin\phi' \neq 2\pi n \end{aligned}\right. \quad \{n=0,1,2...\}$$

$$\Rightarrow kd\sin\phi' = 2\pi\frac{m}{N} \rightarrow \left\{\begin{array}{l}\text{for } m \text{ integer that is not an} \\ \text{integral multiple of } N\end{array}\right\}$$

$\Rightarrow$ Between any adjacent maxima there will be N-1 nulls!

Consider the 1st maxima. The angle for the 1st maxima is given by: $kd\sin\phi_1' = 2\pi$. $(n=1,\ m=N)$

We get a null when the angle is increased from ϕ_1' to $\phi_1' + \frac{\Delta\phi'}{2}$.

The first null after the 1^{st} maxima happens when:

$$kd \sin\left(\phi_1' + \frac{\Delta\phi'}{2}\right) = 2\pi \frac{N+1}{N} \quad (m = N+1).$$

$$\Rightarrow \quad kd \left\{ \sin\phi_1' \cos\frac{\Delta\phi'}{2} + \cos\phi_1' \sin\frac{\Delta\phi'}{2} \right\} = 2\pi\left(1 + \frac{1}{N}\right).$$

~~for $\Delta\phi'$ small { this will happen when N is large }~~

$$kd \left\{ \sin\phi_1' + \cos\phi_1' \frac{\Delta\phi'}{2} \right\} = 2\pi + \frac{2\pi}{N}. \qquad \left\{ \begin{array}{l} \text{recall that:} \\ kd\sin\phi_1' = 2\pi \end{array} \right.$$

$$\Rightarrow \quad kd\cos\phi_1' \frac{\Delta\phi'}{2} = \frac{2\pi}{N}.$$

$$\Rightarrow \quad \Delta\phi' = \frac{2\lambda}{Nd.\cos\phi_1'} \qquad \underline{\text{Ans}}$$

<u>12.4</u>

a) $kd\sin\phi_1' = 2\pi \Rightarrow \sin\phi_1' = \frac{\lambda}{d} \Rightarrow \phi_1' = 23.58°$

$\Rightarrow$ Horizontal displacement $= x_1 = L\tan(\phi_1') = 130.93$ mm

b) Horizontal size of the spot $= L\tan\left(\phi_1' + \frac{\Delta\phi'}{2}\right) - L\tan\left(\phi_1' - \frac{\Delta\phi'}{2}\right)$

$$\left\{ \text{where } \Delta\phi' = \frac{2\lambda}{Nd\cos\phi_1'} \right\} \qquad \approx L\sec^2(\phi_1')\,\Delta\phi'$$

$$\approx \frac{2L\lambda}{Nd} \frac{\sec^2(\phi_1')}{\cos(\phi_1')} = 1.04 \text{ mm}$$

c) Horizontal separation betwen the 1^{st}-order spots of the two wavelengths $= L\tan\left[\sin^{-1}\left(\frac{\lambda_1}{d}\right)\right] - L\tan\left[\sin^{-1}\left(\frac{\lambda_2}{d}\right)\right] = 0.31$ mm.

d) No – the spot size of one wavelength is bigger than the separation between the spots corresponding to wavelengths separated by 1 nm.

e) The best way to improve the resolution is to increase N. Notice that the angular width of the spot is inversely proportional to N. So increasing N will decrease the spot size. The separation between the spots corresponding to the two wavelengths does not change with N. So if we increase N from 300 to 1200, the spot size will be reduced to $\frac{1.04}{4} = 0.26$ mm and the spectrometer will have a wavelength resolution of 1nm