School of Electrical and Computer Engineering, Cornell University

ECE 303: Electromagnetic Fields and Waves

Fall 2007

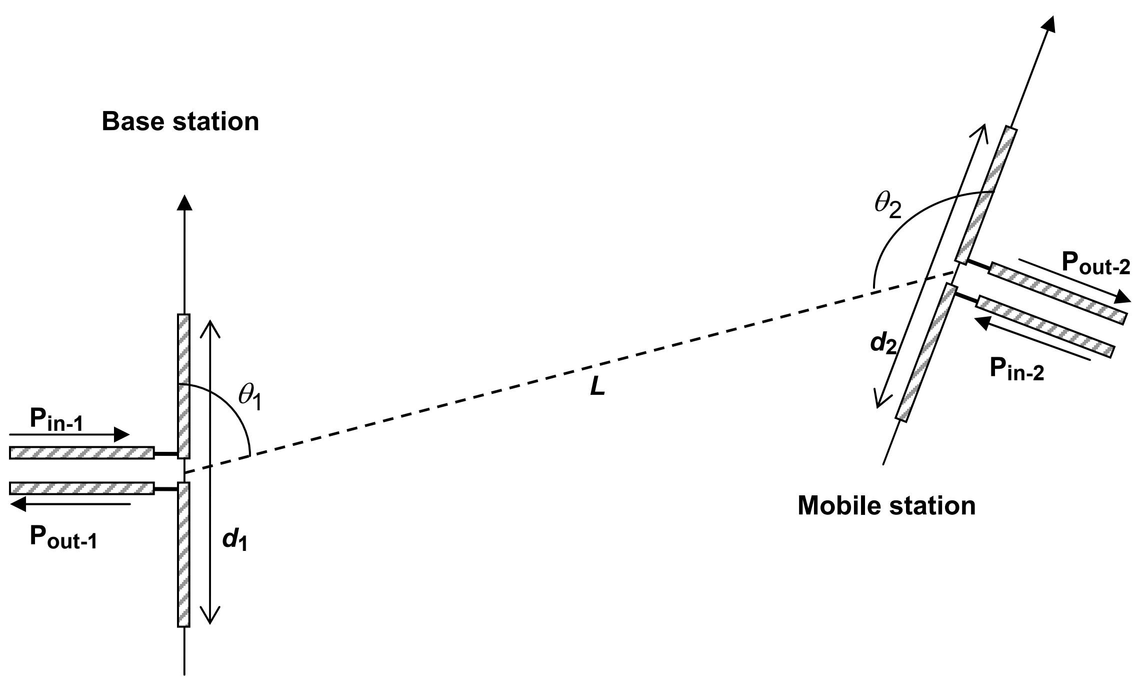
Homework 12 **Due on Nov. 28, 2007 by 5:00 PM**

Reading Assignments:

i) Review the lecture notes.
ii) Review sections 9.1-9.5, 9.7, 9.8 of the paperback book *Electromagnetic Waves*.

Problem 12.1: (A wireless link of two half-wave dipoles)

Consider the following setup for a wireless link (base station and a mobile station) consisting of two **half-wave dipoles**. The powers shown are the NET powers delivered to or received from the antennas.



Assume that:

$\theta_1 = 30^o \quad \theta_2 = 150^o \quad L = 1\text{km} \quad d_1 = d_2 = 15\,\text{cm} \quad f = 1.0\,\text{GHz}$

The antennas are always assumed to be matched to their respective driving/receiving circuits. The total impedance of each antenna was measured and was found to equal: $Z_A = 100 + j43\ \Omega$ (the reactive part of the impedance of an ideal half-wave dipole is always $j43\ \Omega$).

a) What is the dissipative part R_{diss} of the antenna impedance?

b) What is the radiative efficiency η_{rad} of each antenna?

c) What is the gain $G_1(\theta,\phi)$ of the base station antenna in the direction of the mobile station? Give the expression and the numerical value as an answer. Make sure you are using the IEEE definition of the antenna gain.

d) What is the effective area $A_2(\theta,\phi)$ of the mobile station antenna looking in the direction of the base station? Give the expression and the numerical value as an answer.

e) If the base station transmits and the mobile station receives, find the ratio P_{out-2}/P_{in-1}. Give the expression and the numerical value as an answer.

f) If the mobile station transmits and the base station receives, find the ratio P_{out-1}/P_{in-2}. Give the expression and the numerical value as an answer.

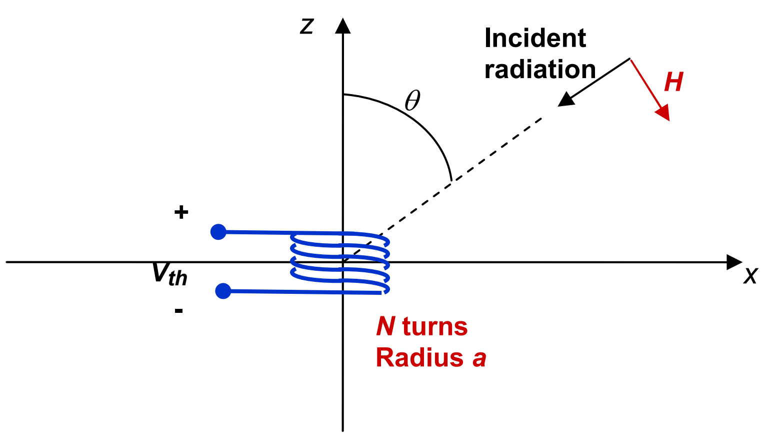
g) Now suppose some engineer made a mistake in designing the mobile station so that the impedance looking into the transmission line from the antenna end is not the matched value: $Z_A^* = 100 - j43\ \Omega$ but is $100\ \Omega$. Find the ratio P_{out-2}/P_{in-1} in this case when the mobile station antenna is not matched to its circuit.

Problem 12.2: (N-turn small wire loop antenna)

Consider the N-turn small wire loop antenna shown below. In this problem you will verify the antenna theorem:

$$A(\theta,\phi) = \frac{\lambda^2}{4\pi} G(\theta,\phi)$$

for the small wire loop antenna in the same way as was done in the lecture notes for a short-dipole antenna. Since the expression for gain has already been derived in the lecture notes, you will need to calculate the antenna effective area directly and then verify the above relation. Consider plane wave radiation with Poynting vector magnitude P incident from the θ direction. You will need to find the power received P_{rec} by a matched circuit connected to the antenna.



a) Find the magnitude $|H|$ of the H-field for the incident plane wave given its Poynting vector magnitude P.

b) Using Faraday's law, find the voltage phasor V_{th} generated across the open-circuited ends of the loop when a plane wave with magnetic field phasor H passes through the loop. Assume that the plane wave is incident from the θ direction and the H-field is polarized in the $\hat{\theta}$-direction (as shown in the figure above). **Hint:** don't forget the angular dependence here. Note that when angle θ is 0-degrees no magnetic flux passes through the loop and when the angle θ is 90-degrees maximum magnetic flux passes through the loop.

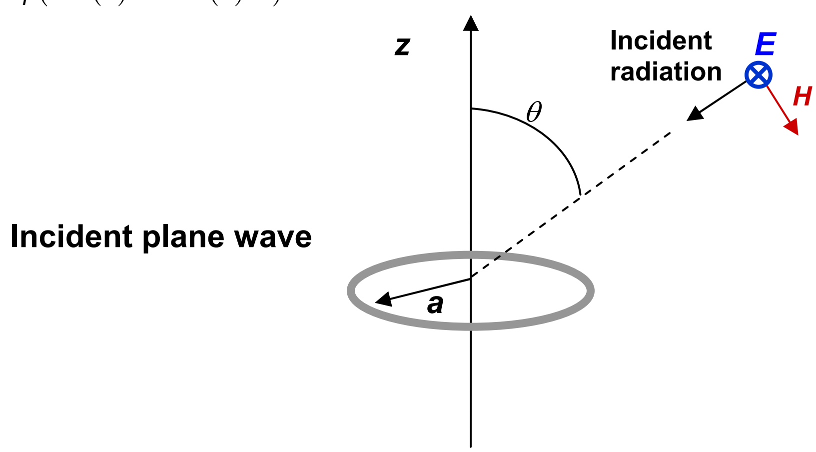
c) From you result in part (b), find the maximum power P_{rec} delivered to a matched load by the antenna. Assume that the impedance of the antenna equals its radiation resistance that was calculated in the lecture notes.

d) Cast your answer in part (c) in terms of the incident power per unit area (i.e. the Poynting vector magnitude P) and find the effective area $A(\theta,\phi)$ of the antenna. Verify that your answer satisfies the antenna theorem.

Problem 12.3: (Electromagnetic scattering from a conducting loop)

Consider a conducting loop of a wire with radius a ($a << \lambda$) and with its axis oriented along the z-axis. The total resistance of the wire loop is R_{diss}. A plane wave is incident on the loop as shown in the figure below. The H-field phasor of the incident plane wave at the location of the loop is given as,

$\vec{H}_i(\vec{r}=0) = H_i\left(\cos(\theta)\hat{x} - \sin(\theta)\hat{z}\right)$



a) Find the phasor I for the current that is induced in the loop due to the incident plane wave.

b) Find the far-field expression $\vec{E}_{s-ff}(\vec{r})$ for the scattered E-field? Think of the scattered field as the field radiated by the induced current in the antenna.

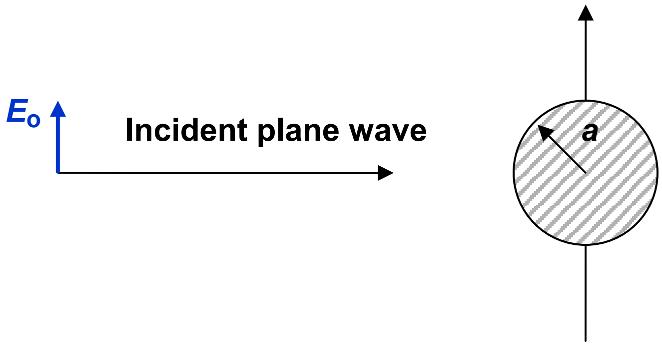
c) Find the total scattered power P_s.

d) Find the scattering cross-section σ_s of the conducting loop.

Problem 12.4: (Electromagnetic scattering from a perfect metal sphere)

Consider a perfect metal sphere with radius a ($a << \lambda$). A plane wave is incident on the sphere as shown in the figure below. The E-field of the incident plane wave at the location of the sphere is given as,

$\vec{E}_i(\vec{r}=0)=\hat{z}E_i$



a) Find the dipole moment phasor $\vec{p}$ ("small p") that is induced in the sphere due to the incident plane wave.

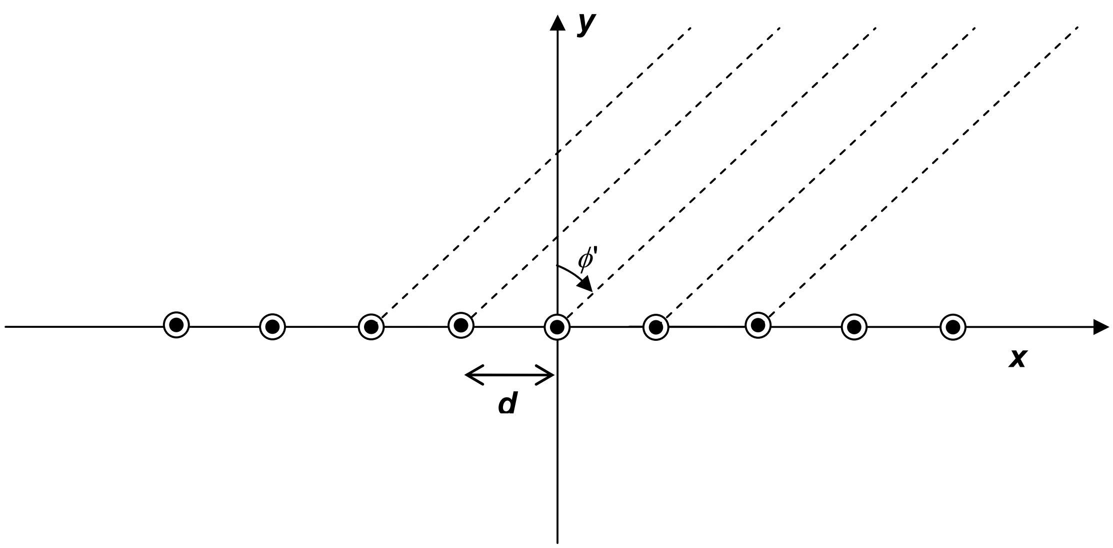
b) Find the far-field expression $\vec{E}_{s-ff}(\vec{r})$ for the scattered E-field?

c) Find the total scattered power P_s.

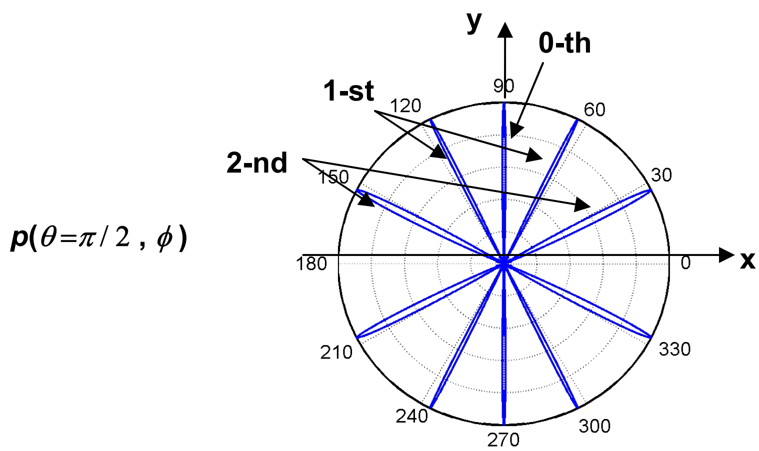
d) Find the scattering cross-section σ_S of the metal sphere and compare it to its actual full cross-sectional area which equals πa^2.

Problem 12.5: (Hertzian dipole linear array)

Consider an array of N Hertzian dipoles separated by distance d and pointing towards the +z-direction. The wavelength of the radiation is λ. All the dipoles have the same currents (magnitudes and phases). The angle ϕ' is defined from the +ve y-axis (as shown above). **The results from this problem will be used in the next problem.**



A very general plot of the radiation pattern is shown below. The specifics of the plot are not related to this problem. The plot is only for illustrative purposes.



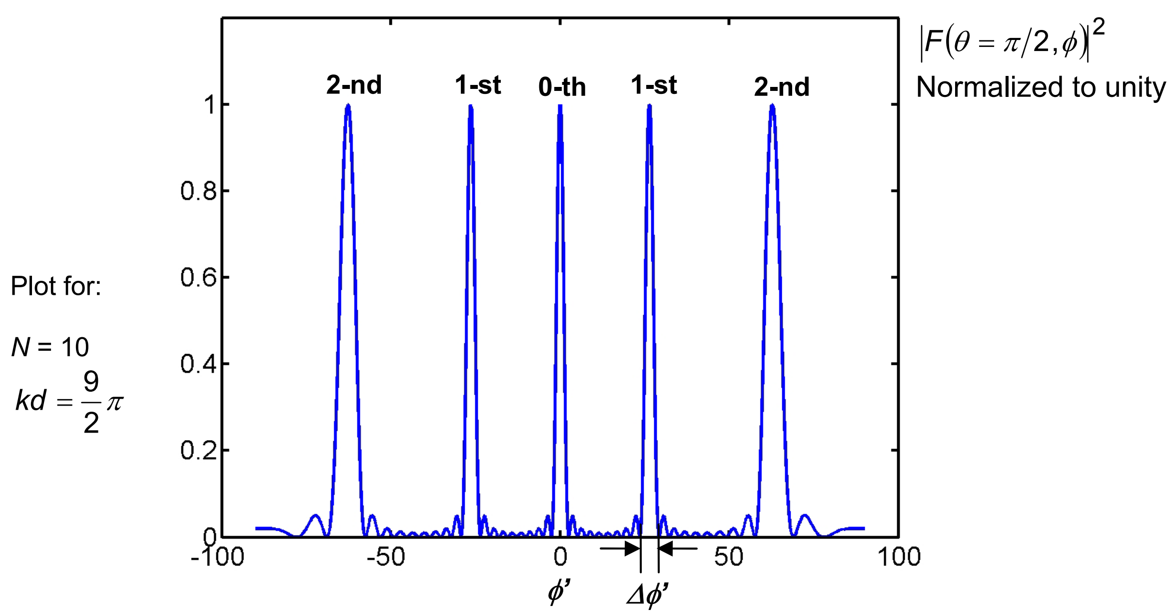
a) Show (based on physical reasoning) that there is a maximum in the radiation pattern for $\phi' = 0$, irrespective of the wavelength λ and the distance d. This is the **0-th order maximum**.

b) Find a condition (without calculating the array factor) in terms of the wavelength λ and the distance d that gives the angles ϕ'_1 for the **1-st order maxima** (see the figure above).

c) Find a condition (without calculating the array factor) in terms of the wavelength λ and the distance d that gives the angles ϕ'_2 for the **2-nd order maxima** (see the figure above).

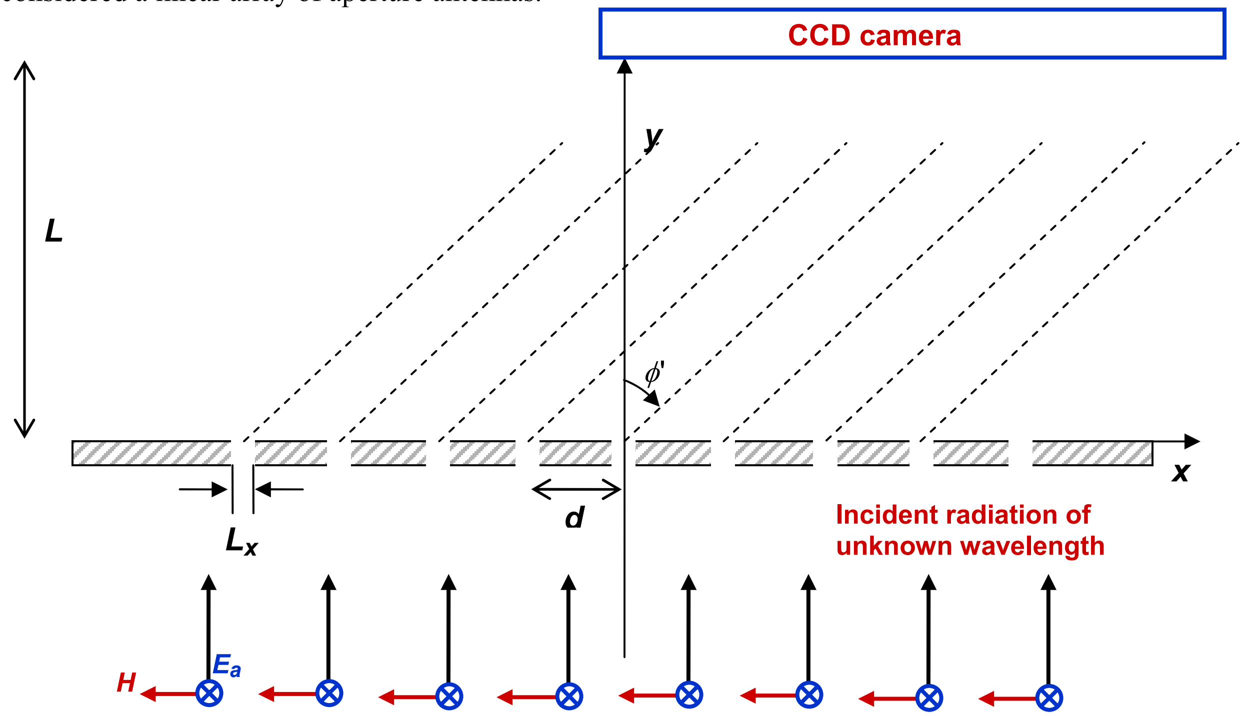
d) Calculate the array factor $F(\theta = \pi/2, \phi)$ for the dipole array and, noting that $\phi' = \pi/2 - \phi$, verify that the angles in parts (b) and (c) do indeed yield maxima in the radiation pattern.

e) Consider the first order maxima. You need to find the angular **full-width** $\Delta\phi'$ of the lobes corresponding to the first order maxima. If you plot the array factor on a linear plot you will get something shown below. Find an expression for the value $\Delta\phi'$ in terms of the parameters N, d, ϕ'_1 and λ.



Problem 12.6: (Diffraction grating spectrometer)

In this problem you will look at the operation of a diffraction grating spectrometer. A diffraction grating consists of N openings (or lines) ruled in a metal coating on a glass substrate (as shown in the figure below). Each opening (or aperture) has width L_x in the x-direction and L_z in the z-direction. The length L_z is very large and could be considered infinite for all practical purposes. So the openings are long and narrow slits. Each opening can be considered an aperture antenna, and so the diffraction grating can be considered a linear array of aperture antennas.



The element factor for this antenna array is just the far-field calculated in the lecture notes for a single rectangular aperture:

$$\vec{E}(r,\theta,\phi) = \vec{E}_{ff}(\vec{r}) = (-\hat{\theta})\frac{j\,k}{2\pi r} E_a \sin(\theta) e^{-j k r} (L_x\ L_z) \frac{\sin(k_x L_x/2)}{k_x L_x/2} \frac{\sin(k_z L_z/2)}{k_z L_z/2}$$

Note that in the above expression the θ,ϕ dependence is embedded in the k_x and k_z terms:

$$k_x = \vec{k}.\hat{x} = k\hat{r}.\hat{x} = k\sin(\theta)\cos(\phi)$$

$$k_z = \vec{k}.\hat{z} = k\hat{r}.\hat{z} = k\cos(\theta)$$

The slit width in the z-direction is very large. For $L_z \to \infty$,

$$\frac{\sin(k_z L_z/2)}{k_z L_z/2} \to \begin{cases} 1 & \text{for} \quad \theta = \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

which means that the output beam does not diffract significantly in the θ-direction (only in the ϕ-direction).

A CCD camera is placed in front of the diffraction grating at a distance L from it. The CCD camera can record the image of the radiation pattern on a flat plane perpendicular to the y-axis.

The spectrometer (as the name suggests) is used to find the spectral content of a light source (i.e. which wavelengths are present in the light from a source). It functions as follows. Plane wave of unknown wavelength is incident upon the grating (as shown in the figure). The radiation coming out from each slit is in phase (since it comes from the same incident plane wave). Radiation coming out from different slits interferes to give a radiation pattern with maxima and minima. The locations of the maxima in the horizontal direction are recorded by the CCD camera. The locations of the maxima in the horizontal direction can be used to figure out the wavelength of the incident radiation since different wavelengths will give maxima for different angles. Usually only the 1-st order maxima are important for this application.

Assume a diffraction grating with the following specifications:

$L_x = 0.1\mu\text{m} \quad d = 1.25\,\mu\text{m} \quad N = 300 \quad L = 300\,\text{mm}$

Suppose light of wavelength 0.5 μm (green light) is incident upon the spectrometer. You need only concern yourself with the maxima and minima in the ($y > 0$, $x > 0$) region. Satisfy yourself that the angular locations of the maxima and minima will be determined completely by the array factor and not the element factor because of the small slit size assumed.

a) Find the angular location of the first order maximum, and find the horizontal displacement (from the y-axis) of the spot (in mm) recorded on the CCD camera due to the 1-st order maximum.

b) Find the horizontal size of the spot (in mm) on the CCD camera due to the 1-st order maximum. Recall that the lobe corresponding to the 1-st order maximum has a finite angular width and this angular width translates into a finite spot size of the image on the CCD camera.

Now consider two different wavelengths, 0.5 μm and 0.501 μm, that are 1 nm apart in wavelength and both are incident at the same time on the spectrometer.

c) Find the horizontal separation (in mm) between the spots on the CCD camera corresponding to the 1-st order maxima for the two wavelengths.

If the spectrometer is to have a wavelength resolution of 1 nm (i.e. it can resolve wavelengths as closely spaced as 1 nm) the spot size calculated in part (b) must be smaller than the separation between the spots of the two wavelengths calculated in part(c). If the spots overlap then the CCD will record one bright spot (as opposed to two distinct spots) and the spectrometer will not be able to resolve the two wavelengths that are 1 nm apart.

d) Based on your calculation in parts (b) and (c) does the spectrometer have a wavelength resolution of 1 nm?

e) What can you do to increase the resolution of the spectrometer? (e.g. change the distance L, change the distance d, change the number of apertures N, etc.).