ECE 303 - Homework #11 Solutions

(By Farhan Rana)

11.1

a) $\omega = \dfrac{2\pi c}{\lambda} = 1.216 \times 10^{15}$ rad/sec.

$$\begin{cases} \varepsilon_1 = \varepsilon_0 n_1^2 \\ \varepsilon_2 = \varepsilon_0 n_2^2 \end{cases}$$

Need to solve:

$$\begin{Bmatrix} \tan(k_x h/2) \\ -\cot(k_x h/2) \end{Bmatrix} = \sqrt{\frac{\omega^2 \mu_0 (\varepsilon_1 - \varepsilon_2)(h/2)^2}{(k_x h/2)^2} - 1}$$

See the attached plot. The values of $k_x h/2$ from the plot come out to be:

TE_1: $k_x \frac{h}{2} = 1.357 \Rightarrow k_x = \frac{2 \times (1.357)}{h} = 2.714 \times 10^6 \ m^{-1}$

TE_2: $k_x \frac{h}{2} = 2.706 \Rightarrow k_x = \frac{2 \times (2.706)}{h} = 5.412 \times 10^6 \ m^{-1}$

TE_3: $k_x \frac{h}{2} = 4.03 \Rightarrow k_x = \frac{2 \times (4.03)}{h} = 8.06 \times 10^6 \ m^{-1}$

$$k_z = \sqrt{\omega^2 \mu_0 \varepsilon_1 - k_x^2}$$

TE_1: $k_z = 1.393 \times 10^7 \ m^{-1}$

TE_2: $k_z = 1.3115 \times 10^7 \ m^{-1}$

TE_3: $k_z = 1.168 \times 10^7 \ m^{-1}$

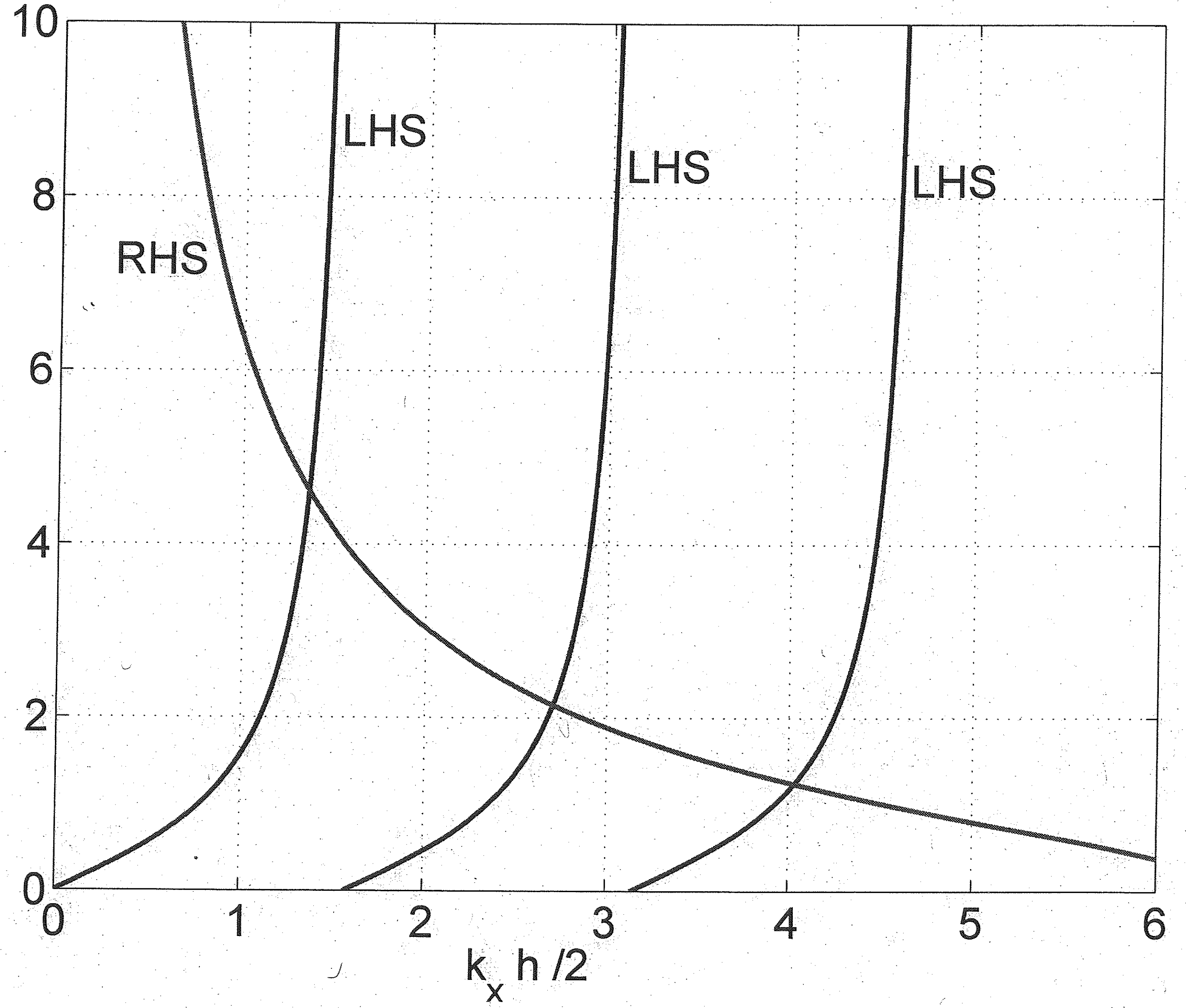
b) TE_1: $n_{eff} = \dfrac{k_z}{\omega/c} = 3.435$

TE_2: $n_{eff} = 3.235$

TE_3: $n_{eff} = 2.880$

c). Cut-off of TE_2 mode is $= \omega_2 = \dfrac{\pi}{h} \dfrac{1}{\sqrt{\mu_0(\varepsilon_1 - \varepsilon_2)}}$

Need $\omega < \omega_2 \Rightarrow h < \dfrac{\pi}{\omega} \dfrac{1}{\sqrt{\mu_0(\varepsilon_1 - \varepsilon_2)}} \Rightarrow h < 0.245 \ \mu m$

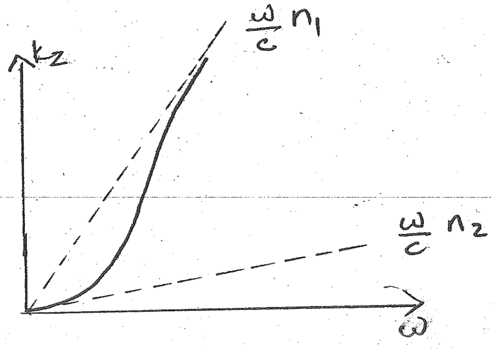


10
8
6
4
2
0
0
1
2
3
4
5
6
RHS
LHS
LHS
LHS
$k_x h/2$

11.2

a) Recall that when $\omega =$ cut-off frequency the angle θ equals θ_c. And when $\omega >$ cut-off frequency then $\theta > \theta_c$ and the wave is guided through total internal reflection. So θ increases with ω. So if $\omega_1 > \omega_2$ then $\theta_1 > \theta_2$.

b) Recall the dispersion curve for TE_1 mode:



Group velocity is the inverse of the slope of the curve, i.e.: $V_g = \frac{d\omega}{dk_z} = \frac{1}{dk_z/d\omega}$

$\Rightarrow$ As $\omega \to 0$ $V_g = \frac{c}{n_2}$ and as $\omega \to \infty$ $V_g = \frac{c}{n_1}$

c) For TM_2 mode:

$$-\cot\left(\frac{k_x h}{2}\right) = \underline{\underline{\frac{n_1^2}{n_2^2}}}\sqrt{\frac{\omega^2 \mu_0 \varepsilon_0 (n_1^2 - n_2^2)}{k_x^2} - 1}$$

For TE_2 mode:

$$-\cot\left(\frac{k_x h}{2}\right) = \sqrt{\frac{\omega^2 \mu_0 \varepsilon_0 (n_1^2 - n_2^2)}{k_x^2} - 1}$$

$\Rightarrow$ TM_2 mode has a larger k_x value and a smaller k_z value.

$\Rightarrow$ $\theta_{TM} < \theta_{TE}$

11.3

a) At $x=0$ need E_y (i.e component parallel to metal surface) to be zero. This condition is already met by the "sine" form of the solution.

At $x = -h/2$:

Continuity of parallel E-field $\Rightarrow$ $-E_0 \sin\left(k_x \frac{h}{2}\right) = E_1$ — ①

Continuity of parallel H-field $\Rightarrow$ $E_0 k_x \cos\left(k_x \frac{h}{2}\right) = \alpha_x E_1$ — ②

Dividing ② by ① : $$-\cot\left(k_x \frac{h}{2}\right) = \frac{\alpha_x}{k_x}$$

but $\alpha_x^2 + k_x^2 = \omega^2 \mu_0 (\varepsilon_1 - \varepsilon_2)$ $\Rightarrow$ $$-\cot\left(k_x \frac{h}{2}\right) = \sqrt{\frac{\omega^2 \mu_0 (\varepsilon_1 - \varepsilon_2)}{k_x^2} - 1}$$

or $$-\cot\left(k_x \frac{h}{2}\right) = \sqrt{\frac{\omega^2 \mu_0 (\varepsilon_1 - \varepsilon_2)(h/2)^2}{(k_x h/2)^2} - 1}$$

b). All anti-symmetric TE modes of the waveguide structure in problem 11.1 are the TE modes of the metal-hybrid waveguide. This is because if you start from the waveguide structure in 11.1 and insert a metal plate in the middle of the waveguide then all the anti-symmetric modes would not be disturbed since they were zero at the location of the metal plate to begin with. For the anti-symmetric TE modes of problem 11.1 the transcendental equation was (from Lecture notes)

$$-\cot(k_x h/2) = \sqrt{\frac{\omega^2 \mu_o (\varepsilon_1 - \varepsilon_2)}{(k_x h/2)^2} (h/2)^2 - 1}$$

and this same equation was derived earlier in part (a).

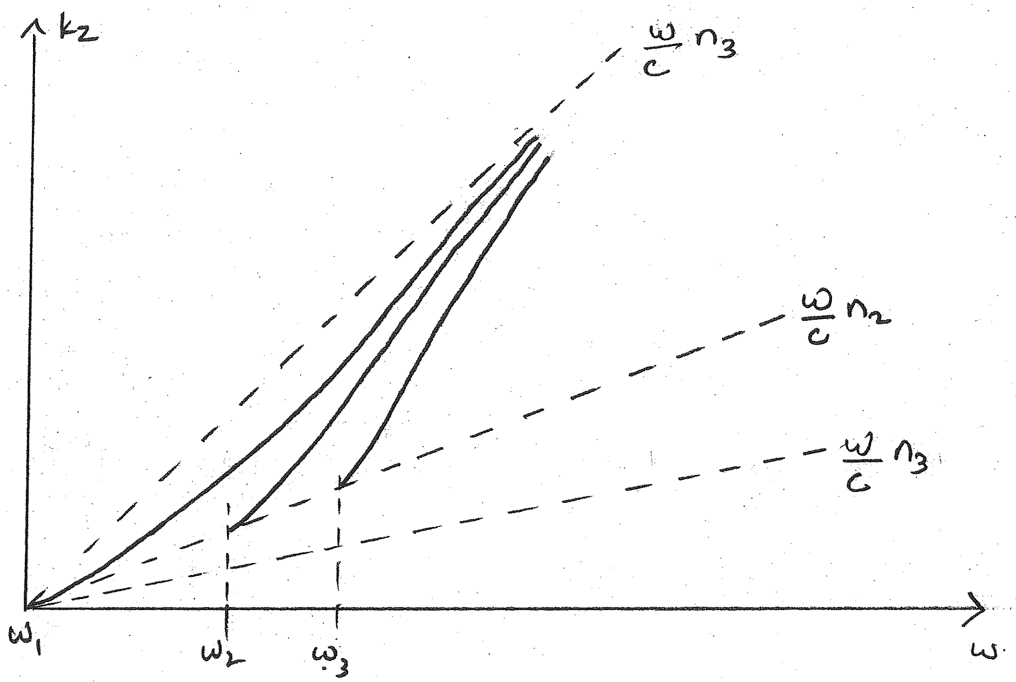
11.4.

The Cut-off frequency corresponds to the condition that the mode is not guided since the wave bouncing around in the core is not totally internally reflected on at least one interface.

Since the critical angle for the Si – Air interface is going to be smaller than the critical angle for the Si – SiO_2 interface (recall that $\sin\theta_c = \frac{n_t}{n_i}$), the total internal reflection at the Si-SiO_2 interface will determine the Cut-off frequency of the guided modes. You may want to read the lecture notes slide with the title "Dielectric Waveguides – What is Cut-off?" to understand this point better.

so $$\omega_m = \frac{(m-1)\pi}{h} \frac{1}{\sqrt{\mu_0(\varepsilon_1-\varepsilon_2)}}$$

b)



$$\begin{cases} n_1 = 3.5 \\ n_2 = 1.5 \\ n_3 = 1.0 \end{cases}$$

Note that at cut-off: $\omega = \omega_m$ and $k_x = (m-1)\frac{\pi}{h}$

$$\Rightarrow \quad k_z = \sqrt{\omega^2 \mu_0 \varepsilon_1 - k_x^2} = \sqrt{\omega_m^2 \mu_0 \varepsilon_1 - k_x^2}$$

$$= \frac{(m-1)\pi}{h} \sqrt{\frac{\mu_0 \varepsilon_1}{\mu_0(\varepsilon_1-\varepsilon_2)} - 1}$$

$$k_z = \frac{(m-1)\pi}{h} \frac{\sqrt{\mu_0 \varepsilon_2}}{\sqrt{\mu_0(\varepsilon_1-\varepsilon_2)}} = \frac{\omega_m}{c} n_2$$

$k_z = \frac{\omega_m}{c} n_2$. That is why in the sketch above, the lines start from the middle dashed curve.

11.5

Following the lecture notes, one obtains:

$$\vec{S}_{ff}(\vec{r})\Big|_{\theta=\frac{\pi}{2}} = \hat{r}\,\eta_o \left|\frac{kId}{4\pi r}\right|^2 \left| e^{j\alpha} e^{jk\frac{h}{2}\cos\phi} + e^{-jk\frac{h}{2}\cos\phi}\right|^2$$

$$\Rightarrow \quad P(\theta=\tfrac{\pi}{2},\phi) = \frac{1+\cos\left[kh\cos(\phi)+\alpha\right]}{2} \qquad \begin{cases} \alpha = \frac{\pi}{2} \\ kh = \pi \end{cases}$$

a) See the plots on the attached page.

b) See the plots on the attached page.

11.6

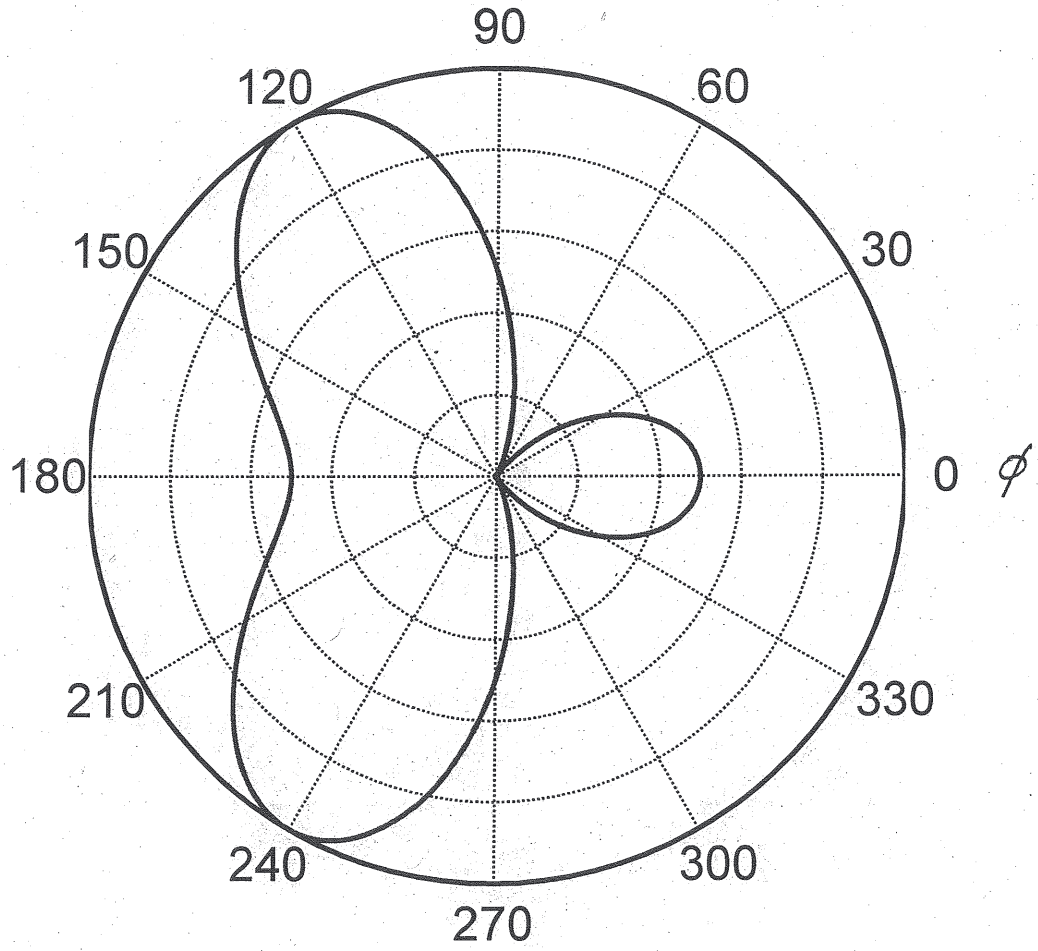
Following the lecture notes:

$$P(\theta,\phi) \propto \frac{\sin^2(\theta)}{2}\left[1+\cos\left(2\pi\cos(\theta)-\pi\right)\right]$$

a) and b) See the attached plots page.

Note: In the plots I have normalized $P(\theta,\phi)$ so that the maximum value is unity.

Problem 11.5 (a)



Problem 11.6 (a)