
ECE 303: Electromagnetic Fields and Waves

Fall 2007

Homework 11

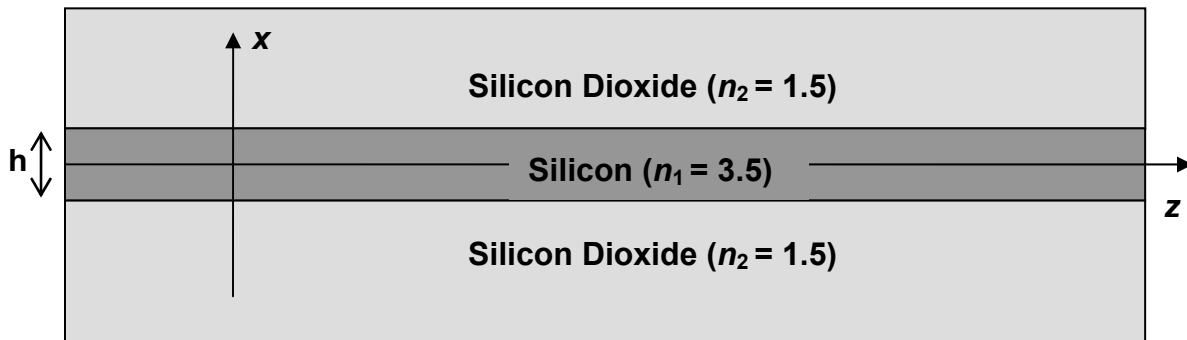
Due on Nov. 9, 2007 by 5:00 PM

Reading Assignments:

- i) Review the lecture notes.
- ii) Review sections 7.1, 7.2, 7.4, and the entire chapter 2 of the paperback book *Electromagnetic Waves*.

Problem 11.1: (Dielectric optical waveguides)

Consider the following integrated dielectric optical waveguide on a Silicon chip:



The dielectric waveguide is designed to carry light whose free-space wavelength is $1.55 \mu\text{m}$ (you can find the corresponding frequency ω from this information).

A side note: $1.55 \mu\text{m}$ is the also wavelength which is used for fiber-optic communications. This wavelength is not in the visible part of the spectrum and so cannot be seen by the eye.

Propagation vectors:

- a) Assuming that $h = 1.0 \mu\text{m}$, find the values of k_z for the first three TE modes of the waveguide for the frequency ω indicated above. **Hint:** You will need to solve a transcendental equation **graphically** to find the value of k_x for each TE mode, and then using the value of k_x find the corresponding value of k_z . Those of you who are thinking of using routines in Matlab or Mathematica to solve transcendental equations need to be careful of the fact that many such routines may not give all the solutions to an equation if multiple solutions exist.

Effective indices:

- b) Usually the value k_z is not specified directly (because it can be a long cumbersome number). Instead the **effective index** n_{eff} defined by the relation,

$$n_{eff} = \frac{k_z}{\omega/c}$$

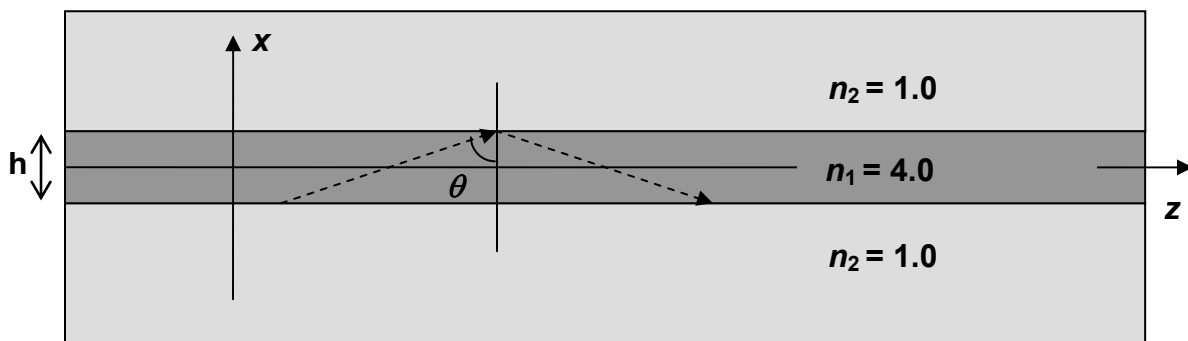
is used to specify k_z . The effective index is the index of a hypothetical medium in which a plane wave of frequency ω would have the same wavevector k_z as that of the waveguide mode. Using your results from part(a), find the effective indices of the first three TE modes. You need to give numerical values as answers.

Single-mode waveguides:

c) In many cases, one wants to have optical waveguides that can propagate only one mode for a given frequency ω . The way to achieve this in practice is to shrink the dimensions of the waveguide core until only the lowest mode can propagate at the given frequency ω and the cut-off frequencies of all the higher modes are larger than ω (note that the cut-off frequencies of modes increase as the waveguide core dimensions are reduced). Assuming the frequency indicated above for light of free-space wavelength $1.55 \mu\text{m}$, find how small the height h of the waveguide ought to be such that only the lowest TE mode (i.e. TE₁ mode) can propagate in the waveguide and all the higher TE modes cannot propagate. Give a numerical answer.

Problem 11.2 (Test your concepts)

Consider a dielectric slab waveguide shown below:



In the figure above, the guided wave angle of incidence θ is also shown.

a) Consider guided TE₂ mode for two different frequencies ω_1 and ω_2 where $\omega_1 > \omega_2$. Will the guided wave angle of incidence θ_1 at frequency ω_1 be larger or smaller than the angle of incidence θ_2 at frequency ω_2 ? Explain your answer. No points will be awarded for wrong or inadequate explanation.

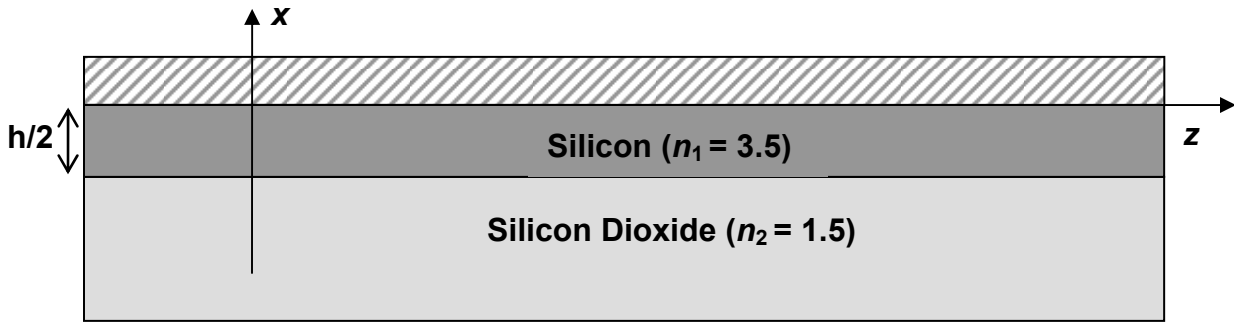
b) Consider guided TE₁ mode. What is the group velocity $v_g = d\omega/dk_z$ of the guided TE₁ mode in the limit when the frequency ω approaches 0. What is the group velocity $v_g = d\omega/dk_z$ of the guided TE₁ mode in the limit when the frequency ω approaches ∞ . Explain your answer. No points will be awarded for wrong or inadequate explanation. Your answer should be in terms of the material and waveguide parameters.

c) Consider guided TE₂ and TM₂ modes for the same frequency ω that is larger than the cut-off frequency of both the modes. Will the angle of incidence θ_{TE} of the TE₂ mode be larger or smaller than

the angle of incidence θ_{TM} of the TM_2 mode? Explain your answer. No points will be awarded for wrong or inadequate explanation.

Problem 11.3: (Metal-dielectric hybrid waveguides)

Consider the following integrated metal-dielectric hybrid waveguide on a Silicon chip:



The propagating mode is guided by reflection from the perfect-metal layer at the top, and by total internal reflection at the bottom interface. The possible expressions for the E-field of the TE-modes are given below:

$$\vec{E}(x, z) \Big|_{-\frac{h}{2} \leq x \leq 0} = \hat{y} E_0 \sin(k_x x) e^{-j k_z z}$$

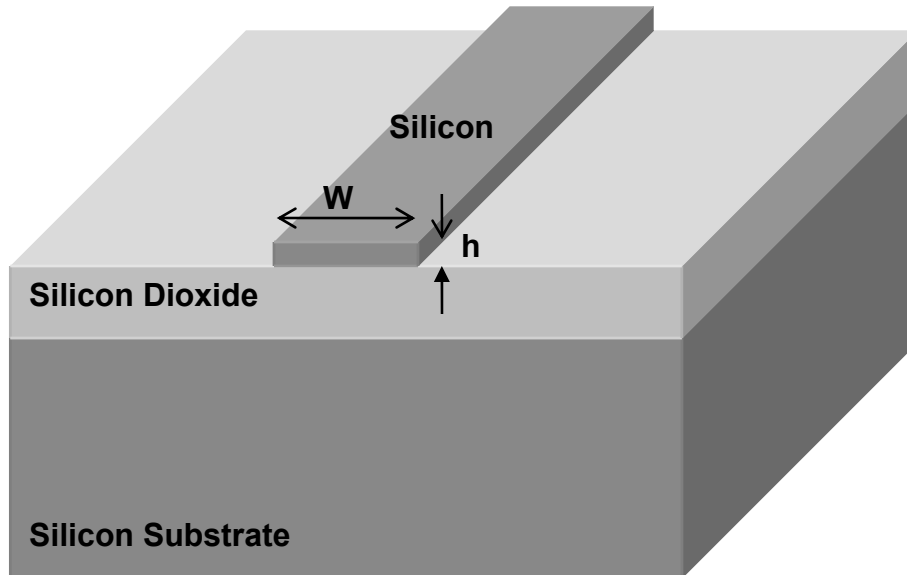
$$\vec{E}(x, z) \Big|_{x \leq -\frac{h}{2}} = \hat{y} E_1 e^{-\alpha_x |x+h/2|} e^{-j k_z z}$$

a) Using the all the boundary conditions at your disposal - at all the interfaces - and the field expressions given above, find a transcendental equation relating k_x to the frequency ω .

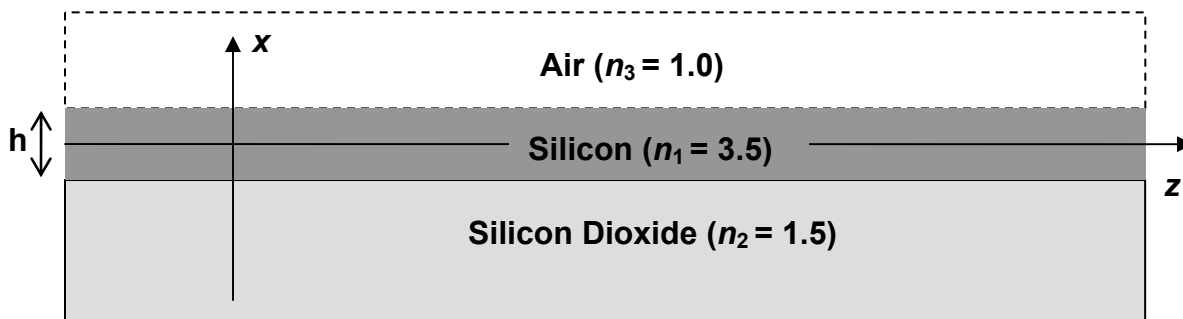
b) It turns out that the transcendental equation for the TE-modes of the above waveguide can also be found in a way that is smarter than what you did in part (a). Consider the waveguide structure of problem 11.1. If you reason carefully you ought to reach a conclusion that all the TE-modes of the above metal-dielectric hybrid waveguide are in fact a subset of the TE-modes of the waveguide in problem 11.1. Can you specify which TE-modes of the waveguide in problem 11.1 correspond to the TE-modes of the above metal-dielectric hybrid waveguide? Using this method of reasoning, find the transcendental equation that relates k_x to the frequency ω for the above metal-dielectric hybrid waveguide and confirm that it is the same as that found in part (a) above.

Problem 11.4: (Asymmetric dielectric optical waveguides)

Consider the following integrated optical waveguide on a Silicon substrate:



As long as the width W of the waveguide is much larger than its height h , the waveguide can be approximated as a slab waveguide shown below:

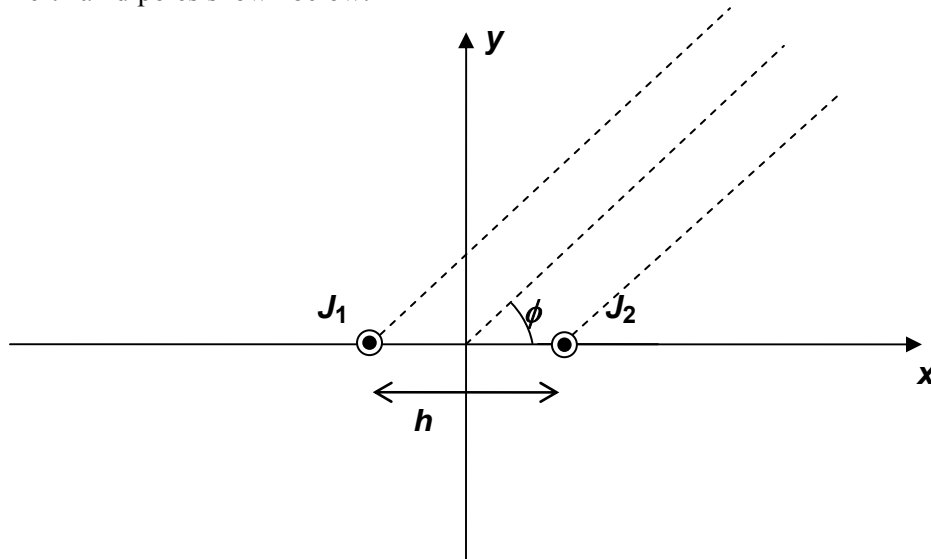


The essential feature about this dielectric waveguide is that it is asymmetric. The cladding layers at top and bottom have different indices. Light is guided in the Silicon core layer. The following two problems will teach you that a lot can be learned about dielectric waveguides without calculating all the details.

- Find the expression for the cut-off frequency ω_m of the TE_m mode? **Hint:** Do not attempt to find the modes or do some elaborate math to get the answer. If you have understood the concepts behind waveguiding in dielectric waveguides you should be able to get the answer in one or two lines without even knowing the expressions for the modes or the transcendental equation that describes the relationship between k_x and ω for the asymmetric waveguide.
- Sketch the dispersion curves (k_z vs ω) for the first three TE modes. Your sketches should also include (with dashed lines) the three dispersion curves for plane waves traveling in media with indices n_1 , n_2 , and n_3 , respectively. Indicate on the sketch the cut-off frequencies for the first three modes as well as the asymptotic behavior of the dispersion curves for large frequencies.

Problem 11.5: (Two Hertzian dipoles)

Consider two Hertzian dipoles shown below:



The current densities of the dipoles are specified by the phasors:

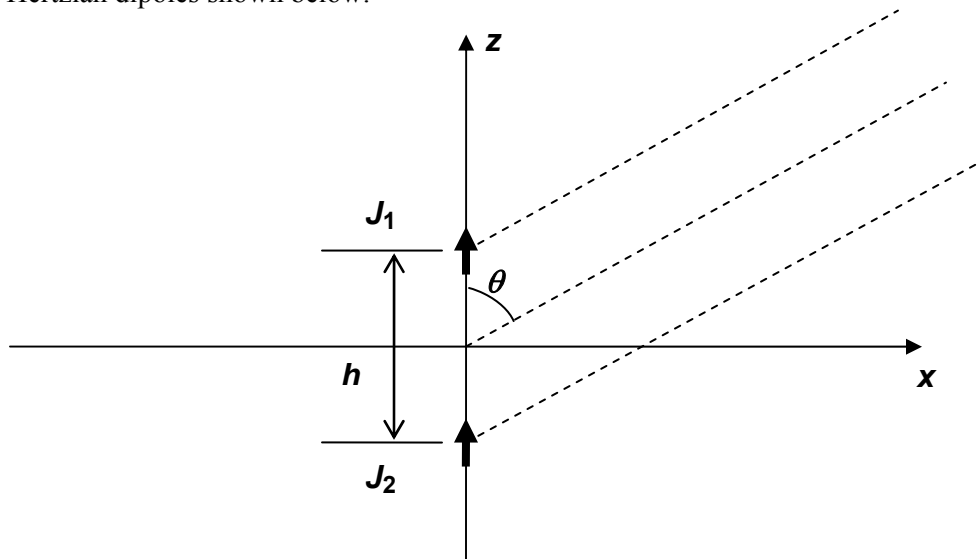
$$\bar{J}_1(\bar{r}) = \hat{z} I d \delta^3\left(\bar{r} + \hat{x} \frac{h}{2}\right)$$

$$\bar{J}_2(\bar{r}) = \hat{z} I d e^{j\alpha} \delta^3\left(\bar{r} - \hat{x} \frac{h}{2}\right)$$

- a) Assuming $\alpha = \pi/2$, $h = \lambda/2$, find the radiation pattern $\rho(\theta = \pi/2, \phi)$ in the y - x plane. Sketch $\rho(\theta = \pi/2, \phi)$ and indicate the angular location of all the nulls and the maxima in the radiation pattern. **The matlab routine “polar” can be used for plotting.**

Problem 11.6: (Two more Hertzian dipoles)

Consider two Hertzian dipoles shown below:



The current densities of the dipoles are specified by the phasors:

$$\bar{J}_1(\vec{r}) = \hat{z} I d \delta^3\left(\vec{r} - \hat{z} \frac{h}{2}\right)$$

$$\bar{J}_2(\vec{r}) = \hat{z} I d A e^{j\alpha} \delta^3\left(\vec{r} + \hat{z} \frac{h}{2}\right)$$

a) Assuming $A = 1$, $\alpha = \pi$, $h = \lambda$, find the radiation pattern $\rho(\theta, \phi = 0)$ in the \mathbf{z} - \mathbf{x} plane. Sketch $\rho(\theta, \phi = 0)$ and indicate the angular location of all the nulls in the radiation pattern. **Hint:** Make sure you do not miss a $\sin^2(\theta)$ term that also comes in due to the angular dependence of the radiation patterns of the individual dipoles.