ECE 303 Homework #7 Solutions

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P.7.1

$\vec{E}_i(\vec{r}) = \hat{x} E_0 e^{-jk_i z}$ $\quad f = 1\,GHz \quad \sigma = 10\,S/m$

At this frequency $\frac{\sigma}{\omega \varepsilon_0} \simeq 180 >> 1$ so the metal alloy is a good conductor

so $k_t = \frac{1-j}{\delta}$ where $\delta = \sqrt{\frac{2}{\omega_0 \mu \sigma}} \approx 0.005\,m$ and $k_i = \frac{2\pi}{\lambda}$

a) Reflection coefficient:

$$\Gamma = \frac{\frac{k_i}{k_t} - 1}{\frac{k_i}{k_t} + 1} = \frac{1 - \frac{k_t}{k_i}}{1 + \frac{k_t}{k_i}} = \frac{1 - \frac{\lambda}{2\pi\delta}(1-j)}{1 + \frac{\lambda}{2\pi\delta}(1-j)}$$

or:

$$\Gamma = \frac{\left(1 - \frac{\lambda}{2\pi\delta}\right) + j\frac{\lambda}{2\pi\delta}}{\left(1 + \frac{\lambda}{2\pi\delta}\right) - j\frac{\lambda}{2\pi\delta}} \quad \text{and: } |\Gamma| = \frac{\left(1 - \frac{\lambda}{2\pi\delta}\right)^2 + \left(\frac{\lambda}{2\pi\delta}\right)^2}{\left(1 + \frac{\lambda}{2\pi\delta}\right)^2 + \left(\frac{\lambda}{2\pi\delta}\right)^2} \simeq \underline{0.9}$$

b) Incident time-averaged power per unit area: $\langle S_i \rangle = \frac{E_0^2}{2\eta_i} = 1\,W/m^2$

Reflected >> >> >> : $\langle S_r \rangle = \frac{E_0^2}{2\eta_i} |\Gamma|^2$

Power per unit area dissipated in metal alloy:

$$\langle S_t \rangle = \langle S_i \rangle - \langle S_r \rangle = \frac{E_0^2}{2\eta_i}\left(1 - |\Gamma|^2\right) = 0.19\,W/m^2$$

Fraction of incident power per unit area dissipated: $\frac{\langle S_t \rangle}{\langle S_i \rangle} = 1 - |\Gamma|^2 = 0.19$

or 19%

P 7. 1 cont'd

c) Transmission coefficient: $T = 1 + \Gamma \Rightarrow T = \dfrac{2}{1 + \frac{\lambda}{2\pi\delta} - j\frac{\lambda}{2\pi\delta}}$

$$|T|^2 = \frac{4}{\left(1+\frac{\lambda}{2\pi\delta}\right)^2 + \left(\frac{\lambda}{2\pi\delta}\right)^2} \simeq 0.02$$

Transmitted E-field: $\vec{E}_t(\bar{r}) = \hat{x}\, T E_0\, e^{-jk_t z}$

Current density in alloy: $\vec{J}(\bar{r}) = \hat{x}\, \sigma T E_0\, e^{-jk_t z}$

$$\frac{1}{2}\mathrm{Re}\left[\vec{J}(\bar{r}) \cdot \vec{E}_t^*(\bar{r})\right] = \frac{1}{2}\sigma |T|^2 E_0^2\, e^{-2k_t'' z} \qquad \text{where: } k_t'' = \frac{2}{\delta} \ (m^{-1})$$

$$= \sigma |T|^2 \eta_i\, e^{-\frac{2}{\delta}z} \qquad \text{and } \frac{E_0^2}{2\eta_i} = 1\,W/m^2 \Rightarrow E_0 = \sqrt{2\eta_i}\ (V/m)$$

$$\Rightarrow \int_0^\infty \frac{1}{2}\mathrm{Re}\left[\vec{J}(\bar{r}) \cdot \vec{E}_t^*(\bar{r})\right] dz = \frac{\delta\sigma\eta_i}{2}|T|^2 \simeq \underline{0.19\ W/m^2} \quad \text{agrees with b)}$$

P 7.2

Reflection off plasma: $n_i = 1 \quad n_t = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$

$$\Gamma = \frac{1 - \frac{n_t}{n_i}}{1 + \frac{n_t}{n_i}}$$

If $\omega < \omega_p$ then $1 - \frac{\omega_p^2}{\omega^2} < 0$

and $n_t = -j\sqrt{\frac{\omega_p^2}{\omega^2} - 1}$ imaginary.

so:

$$\Gamma = \frac{1 + j\sqrt{\frac{\omega_p^2}{\omega^2} - 1}}{1 - j\sqrt{\frac{\omega_p^2}{\omega^2} - 1}} \Rightarrow |\Gamma|^2 = \frac{1 + \left(\frac{\omega_p^2}{\omega^2} - 1\right)}{1 + \left(\frac{\omega_p^2}{\omega^2} - 1\right)} = 1$$

all incident power gets reflected.

P7.3

$$\bar{\bar{\varepsilon}} = \varepsilon_0 \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

refractive index in uniaxial medium: $n_x = n_z = 3$, $n_y = 2$

a) $\vec{E}_i(\bar{r}) = \hat{x} E_0 e^{-jk_i z}$ $\qquad \vec{E}_r(\bar{r}) = \hat{x} \Gamma_x E_0 e^{+jk_i z}$

$$\Gamma_x = \frac{1 - \frac{n_x}{n_i}}{1 + \frac{n_x}{n_i}} = \frac{1-3}{1+3} = -\frac{1}{2}$$

b) $\vec{E}_i(\bar{r}) = \hat{y} E_0 e^{-jk_i z}$ $\qquad \vec{E}_r(\bar{r}) = \hat{y} \Gamma_y E_0 e^{+jk_i z}$

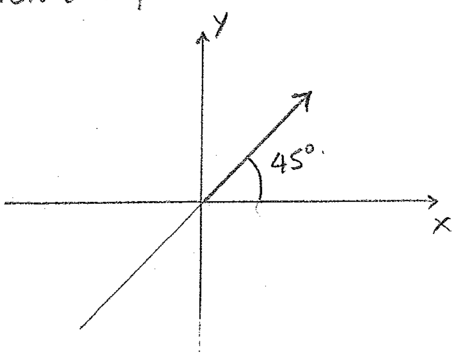
$$\Gamma_y = \frac{1 - \frac{n_y}{n_i}}{1 + \frac{n_y}{n_i}} = \frac{1-2}{1+2} = -\frac{1}{3}$$

c) $E_i(\bar{r}) = \frac{E_0}{2}(\hat{x} + \hat{y}) e^{-jk_i z}$ $\qquad \vec{E}_r(\bar{r}) = \frac{E_0}{2} e^{+jk_i z} (\hat{x}\Gamma_x + \hat{y}\Gamma_y) =$

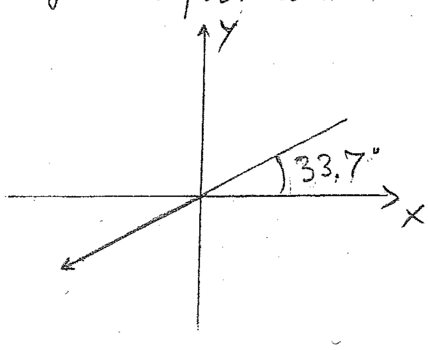
$$= -\frac{E_0}{4} e^{+jk_i z}\left(\hat{x} + \frac{2}{3}\hat{y}\right) \qquad \operatorname{atan}\left(\tfrac{2}{3}\right) \simeq 33.7^\circ$$

at $z = 0$:

incident polarization:



reflected polarization:



Polarization remains linear but its axis is rotated by 11.3°

P 7.4

a) Incident fields: $\vec{E}_i = \hat{y} E_{iy} e^{-j\vec{k}_i \cdot \vec{r}}$, $\vec{H}_i = (\hat{x} H_{ix} + \hat{z} H_{iz}) e^{-j\vec{k}_i \cdot \vec{r}}$

where: $\vec{k}_i = \frac{\omega}{c} n_i \left(\cos(\theta_i) \hat{z} + \sin(\theta_i) \hat{x} \right)$

$$\vec{H}_i = \frac{\vec{k}_i \times \vec{E}_i}{\omega \mu_0} = \frac{E_{iy}}{\eta_i} \left(-\cos(\theta_i) \hat{x} + \sin(\theta_i) \hat{z} \right) e^{-j\vec{k}_i \cdot \vec{r}} \Rightarrow \begin{cases} H_{ix} = -\frac{E_{iy}}{\eta_i} \cos(\theta_i) \\ H_{iz} = \frac{E_{iy}}{\eta_i} \sin(\theta_i) \end{cases}$$

Reflected fields: $\vec{E}_r = \hat{y} E_{ry} e^{-j\vec{k}_r \cdot \vec{r}}$, $\vec{H}_r = (\hat{x} H_{rx} + \hat{z} H_{rz}) e^{-j\vec{k}_r \cdot \vec{r}}$

where: $\vec{k}_r = \frac{\omega}{c} n_i \left(-\cos(\theta_i) \hat{z} + \sin(\theta_i) \hat{x} \right)$ since $\underline{\theta_r = \theta_i}$

$$\vec{H}_r = \frac{\vec{k}_r \times \vec{E}_r}{\omega \mu_0} = \frac{E_{ry}}{\eta_i} \left(\cos(\theta_i) \hat{x} + \sin(\theta_i) \hat{z} \right) e^{-j\vec{k}_r \cdot \vec{r}} \Rightarrow \begin{cases} H_{rx} = \frac{E_{ry}}{\eta_i} \cos(\theta_i) \\ H_{rz} = \frac{E_{ry}}{\eta_i} \sin(\theta_i) \end{cases}$$

Transmitted fields: $\vec{E}_t = \hat{y} E_{ty} e^{-j\vec{k}_t \cdot \vec{r}}$, $\vec{H}_t = (\hat{x} H_{tx} + \hat{z} H_{tz}) e^{-j\vec{k}_t \cdot \vec{r}}$

where: $\vec{k}_t = \frac{\omega}{c} n_t \left(\cos(\theta_t) \hat{z} + \sin(\theta_t) \hat{x} \right)$ and: $\sin(\theta_t) = \frac{n_i}{n_t} \sin(\theta_i)$

$$\vec{H}_t = \frac{\vec{k}_t \times \vec{E}_t}{\omega \mu_0} = \frac{E_{ty}}{\eta_t} \left(-\cos(\theta_t) \hat{x} + \sin(\theta_t) \hat{z} \right) e^{-j\vec{k}_t \cdot \vec{r}} \Rightarrow \begin{cases} H_{tx} = -\frac{E_{ty}}{\eta_t} \cos(\theta_t) \\ H_{tz} = \frac{E_{ty}}{\eta_t} \sin(\theta_t) \end{cases}$$

i) $$\frac{H_{rx}}{H_{ix}} = \frac{E_{ry}}{\eta_i} \cos(\theta_i) \frac{1}{-\frac{E_{iy}}{\eta_i} \cos(\theta_i)} = -\frac{E_{ry}}{E_{iy}} \overset{(1)}{=\!=} \frac{1 - \eta_t \cos(\theta_i) / \eta_i \cos(\theta_t)}{1 + \eta_t \cos(\theta_i) / \eta_i \cos(\theta_t)}$$

ii) $$\left. \frac{H_{tz}}{H_{iz}} = \frac{E_{ty} \sin(\theta_t)}{\eta_t} \frac{1}{\frac{E_{iy}}{\eta_i} \sin(\theta_i)} = \frac{E_{ty}}{E_{iy}} \frac{\frac{1}{\eta_t} \sin(\theta_t)}{\frac{1}{\eta_i} \sin(\theta_i)} \right\} \Rightarrow \frac{H_{tz}}{H_{iz}} = \frac{E_{ty}}{E_{iy}} \overset{(1)}{=\!=} \frac{2 \frac{\eta_t \cos(\theta_i)}{\eta_i \cos(\theta_t)}}{1 + \frac{\eta_t \cos(\theta_i)}{\eta_i \cos(\theta_t)}}$$

but: $n_i \sin(\theta_i) = n_t \sin(\theta_t) \Rightarrow \frac{1}{\eta_i} \sin(\theta_i) = \frac{1}{\eta_t} \sin(\theta_t)$

P7.4 cont'd   we use expressions for $\vec{k}_{i,r,t}$ from (a)

b) Incident fields: $\vec{H}_i = \hat{y} H_{iy} e^{-j\vec{k}_i\cdot\vec{r}}$, $\vec{E}_i = (\hat{x}E_{ix} + \hat{z}E_{iz}) e^{-j\vec{k}_i\cdot\vec{r}}$

$$\vec{E}_i = -\frac{\vec{k}_i \times \vec{H}_i}{\omega \varepsilon_0} = -\eta_i H_{iy}\left(-\cos(\theta_i)\hat{x} + \sin(\theta_i)\hat{z}\right) e^{-j\vec{k}_i\cdot\vec{r}} \Rightarrow \begin{cases} E_{ix} = \eta_i H_{iy}\cos(\theta_i) \\ E_{iz} = -\eta_i H_{iy}\sin(\theta_i) \end{cases}$$

Reflected fields: $\vec{H}_r = \hat{y} H_{ry} e^{-j\vec{k}_r\cdot\vec{r}}$, $\vec{E}_r = (\hat{x}E_{rx} + \hat{z}E_{rz}) e^{-j\vec{k}_r\cdot\vec{r}}$

$$\vec{E}_r = -\frac{\vec{k}_r \times \vec{H}_r}{\omega \varepsilon_0} = -\eta_i H_{ry}\left(\cos(\theta_i)\hat{x} + \sin(\theta_i)\hat{z}\right) e^{-j\vec{k}_r\cdot\vec{r}} \Rightarrow \begin{cases} E_{rx} = -\eta_i H_{ry}\cos(\theta_i) \\ E_{rz} = -\eta_i H_{ry}\sin(\theta_i) \end{cases}$$

Transmitted fields: $\vec{H}_t = \hat{y} H_{ty} e^{-j\vec{k}_t\cdot\vec{r}}$, $\vec{E}_t = (\hat{x}E_{tx} + \hat{z}E_{tz}) e^{-j\vec{k}_t\cdot\vec{r}}$

$$\vec{E}_t = -\frac{\vec{k}_t \times \vec{H}_t}{\omega \varepsilon_0} = -\eta_t H_{ty}\left(-\cos(\theta_t)\hat{x} + \sin(\theta_t)\hat{z}\right) e^{-j\vec{k}_t\cdot\vec{r}} \Rightarrow \begin{cases} E_{tx} = \eta_t H_{ty}\cos(\theta_t) \\ E_{tz} = -\eta_t H_{ty}\sin(\theta_t) \end{cases}$$

i) $$\frac{E_{rz}}{E_{iz}} = \frac{-\eta_i H_{ry}\sin(\theta_i)}{-\eta_i H_{iy}\sin(\theta_i)} = \frac{H_{ry}}{H_{iy}} \overset{(2)}{=\!=} \frac{\eta_i\cos(\theta_i)/\eta_t\cos(\theta_t) - 1}{\eta_i\cos\theta_i/\eta_t\cos\theta_t + 1}$$

i) $$\frac{E_{tx}}{E_{ix}} = \frac{\eta_t H_t\cos(\theta_t)}{\eta_i H_{iy}\cos(\theta_i)} \overset{(2)}{=\!=} \frac{2}{\eta_i\cos(\theta_i)/\eta_t\cos(\theta_t) + 1}$$