ECE 303 Homework #6 Solutions

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P 1.1

$\vec{E}(\vec{r}) = \hat{x} E_0 e^{-jkz}$

We need to show that:

$$-\frac{1}{2} Re\left[\nabla \cdot \vec{S}(\vec{r})\right] = \frac{1}{2} Re\left[\vec{J}(\vec{r}) \cdot \vec{E}^*(\vec{r})\right]$$

Since the medium is lossy k is complex:

$k = k' - jk''$, $k'' > 0$

Dispersion relation: $k = \omega\sqrt{\mu_0 \varepsilon}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{1/2}$

$$\Rightarrow k^2 = \omega^2 \mu_0 \varepsilon - j\omega\mu_0\sigma$$

But $k^2 = (k' - jk'')^2 = k'^2 - k''^2 - j2k'k''$

Equating real and imag. parts:

$k'^2 - k''^2 = \omega^2 \mu_0 \varepsilon$

$2k'k'' = \omega\mu_0\sigma$ (1)

$\vec{S}(\vec{r}) = \vec{E} \times \vec{H}^*$

where:

$$\vec{H} = \frac{k}{\omega\mu_0}(\hat{z} \times \hat{x}) E_0 e^{-jkz} = \hat{y}\frac{(k' - jk'')}{\omega\mu_0} E_0 e^{-jkz}.$$

So $\vec{S}(\vec{r}) = \hat{z}\dfrac{(k' + jk'')}{\omega\mu_0} E_0^2 e^{-2k''z}$

and:

$$-\frac{1}{2} Re\left\{\nabla \cdot \vec{S}(\vec{r})\right\} = -\frac{1}{2} Re\left\{\frac{\partial}{\partial z} S(\vec{r})\right\} = \frac{k'k''}{\omega\mu_0} E_0^2 e^{-2k''z} \overset{(1)}{=} \underline{\sigma E_0^2 e^{-2k''z}}$$

Also:

$\vec{J}(\vec{r}) = \sigma\vec{E}(\vec{r}) = \hat{x}\sigma E_0 e^{-jkz}$

$$-\frac{1}{2} Re\left\{\vec{J}(\vec{r}) \cdot \vec{E}^*(\vec{r})\right\} = \underline{\sigma E_0^2 e^{-2k''z}} = -\frac{1}{2} Re\left\{\nabla \cdot \vec{S}(\vec{r})\right\}$$

P.2.2

$\sigma = 5\times10^{-3}$ S/m $\quad \varepsilon = 10\,\varepsilon_0 \quad f_1 = 10$ KHz $\quad f_2 = 100$ MHz

For f_1: $\quad \omega_1 \tau_d = \frac{\omega_1 \varepsilon}{\sigma} = \frac{2\pi\, 10\times10^{3}\,\mathrm{Hz}\cdot 10\cdot 8.8542\times10^{-12}\,\mathrm{F/m}}{5\times10^{-3}\,\mathrm{S/m}} \simeq 0.0011$

For f_2: $\quad \omega_2 \tau_d = \omega_2 \frac{\varepsilon}{\sigma} = \frac{2\pi\, 100\times10^{6}\,\mathrm{Hz}\cdot 10\cdot 8.8542\times10^{-12}\,\mathrm{F/m}}{5\times10^{-3}\,\mathrm{S/m}} \simeq 11.1$

$\omega_1 \tau_d << 1$: good conductor at 10kHz

$\omega_2 \tau_d >> 1$: "bad" conductor at 100 MHz

b) At f_1: since soil is a good conductor we take the approximation:

$$k = \omega_1\sqrt{\mu_0 \varepsilon}\left(1 - j\frac{\sigma}{\omega_1 \varepsilon}\right)^{1/2} \simeq \omega_1\sqrt{\mu_0\varepsilon}\sqrt{\frac{\sigma}{\omega_1\varepsilon}}\,\frac{1-j}{\sqrt{2}} = \sqrt{\frac{\omega_1\mu_0\sigma}{2}}\,(1-j)$$

$$\Rightarrow k'' = \sqrt{\frac{\omega_1 \mu_0 \sigma}{2}} = \frac{1}{\delta_1} \quad \text{where } \delta_1 \simeq 71.176 \text{ m}$$

Time-averaged power decays as $e^{-2k''z}$ so we need:

$$e^{-2k''z} = 0.01 \Rightarrow z = -\frac{\delta_1}{2}\ln(0.01) \simeq \underline{163.89 \text{ m}}$$

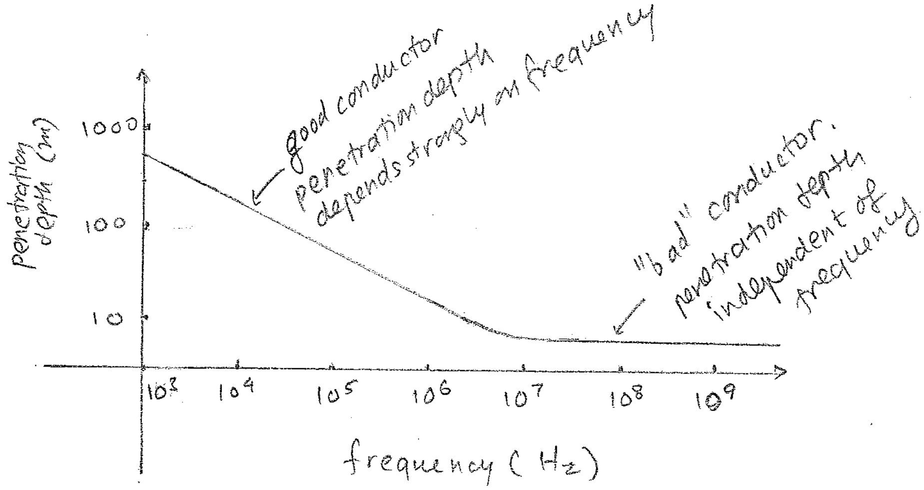
At f_2: since soil is a "bad" conductor we take the approximation:

$$k = \omega_2\sqrt{\mu_0\varepsilon}\left(1 - j\frac{\sigma}{\omega_2 \varepsilon}\right)^{1/2} \simeq \omega_2\sqrt{\mu_0\varepsilon}\left(1 - j\frac{\sigma}{2\omega_2\varepsilon}\right) \Rightarrow k'' = \frac{\sigma\,\eta_0}{2\sqrt{10}} = \frac{1}{\delta_2}$$

where $\delta_2 \simeq 3.35$ m

$$e^{-2k''z} = 0.01 \Rightarrow z = -\frac{\delta_2}{2}\ln(0.01) = \underline{7.71 \text{ m}}$$

c) If we plot penetration depth as a function of frequency in a log-log scale we get:



d) To image an object at depth of 50 m, we need a penetration depth of 100 m to account for the roundtrip of the electromagnetic wave.

From the graph above, we see that for 100m we need a frequency in the "good conductor" region, so we use $k'' = \sqrt{\frac{\omega \mu_0 \sigma}{2}}$

$$e^{-2k''z} = 0.01 \Rightarrow k'' = -\frac{\ln(0.01)}{2z} \Rightarrow f = \frac{[\ln(0.01)]^2}{4z^2} \frac{1}{\pi \mu_0 \sigma}$$

$\Rightarrow f = 2.686 \times 10^4$ Hz or $\underline{26.86 \text{ KHz}}$

The wavelength at this frequency is $\lambda = 1.117$ Km!
So we can't really image any small objects at 50 m depth.

P. 6.3

$N = 10^{12}\ m^{-3}$
$m = 9.1\times10^{-31}\ kg$
$e = 1.6\times10^{-19}\ Cb$

Plasma frequency: $\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}} = 5.637\times10^{7}$ rads/sec

or $f_p = 8.971$ MHz

For the signal to reach the probe, $k = \omega\sqrt{\mu_0\varepsilon_0}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$ must not be imaginary, otherwise no power can be transmitted through the ionosphere.

i.e. we need $1 - \frac{\omega_p^2}{\omega^2} \geq 0 \Rightarrow \omega \geq \omega_p$ or $f \geq f_p = 8.971$ MHz

P. 6.4

$$\bar{\bar{\varepsilon}} = \varepsilon_0 \begin{bmatrix} 4 & & \\ & 4 & \\ & & 9 \end{bmatrix}$$

$\lambda = 0.5\times10^{-6}$ m

a) z-axis is the extra-ordinary axis
x and y-axes are the ordinary axes

b) Since $\varepsilon^e > \varepsilon^o$ the extraordinary axis is the slow axis and the ordinary axes are the fast axes.

c) For propagation in the +y direction, RIGHT-HANDED circular polarization means that the output field vector must have the form:

$$\vec{E} = \underset{\substack{\downarrow \\ \text{complex (V/m)}}}{E_0}\left(\hat{z} - j\hat{x}\right)$$

For simplicity let's assume $|E_0| = 1$ V/m

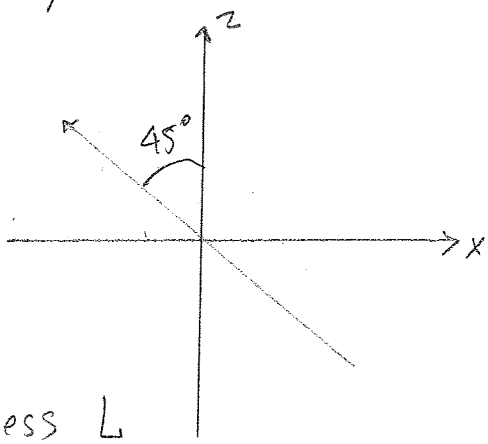
P.6.4

c) cont'd: To achieve this with minimum wave-plate thickness, the field at $0 \le y \le L$ must be:

$$\vec{E} = \hat{z}\, e^{-jk^e y} - \hat{x}\, e^{-jk^o y} =$$

$$= e^{-jk^e y}\left(\hat{z} - \hat{x}\, e^{j(k^e - k_o)y}\right)$$

At $y = 0$ $\vec{E} = \hat{z} - \hat{x}$ i.e.



the wave-plate must be rotated so that the incident linear polarization is at 45° w.r.t. the extra-ordinary ($\hat{z}$) axis of the wave-plate, and its thickness L is such that:

$$(k^e - k^o)L = \frac{\pi}{2} \Rightarrow L = \frac{\pi}{2\frac{2\pi}{\lambda}\left(\sqrt{\varepsilon^e} - \sqrt{\varepsilon^o}\right)} \Rightarrow L = \frac{\lambda}{4} = 0.125 \times 10^{-6}\, m$$

d) For LEFT-HANDED circular polarization the field at the output must be $\vec{E} = E_o\left(\hat{z} + j\hat{x}\right)$ For simplicity assume $|E_o| = 1$ V/m

E_o ↓ complex (V/m)

We could achieve this with the same input as in c) and $L = \frac{3\pi}{2(k^e - k^o)} = \frac{3\lambda}{4}$ but that would not be the minimum L.

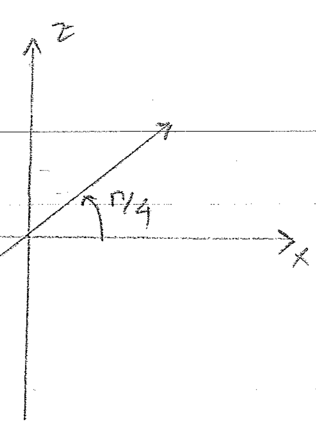
Instead choose a field: $\vec{E} = \hat{z}\, e^{-jk^e y} + \hat{x}\, e^{-jk^o y} =$

$$= e^{-jk^e y}\left(\hat{z} + \hat{x}\, e^{+j(k^e - k^o)y}\right)$$

so again we need $L = \frac{\pi}{2(k^e - k^o)} = \frac{\lambda}{4} = 0.125 \times 10^{-6}\, m$

P 6.4 d) cont'd

At $y=0$ $\quad \vec{E} = \hat{z} + \hat{x}$ i.e.



The wave plate must be rotated so that the incident linear polarization is at 45° w.r.t. x-axis of the waveplate, and the wave-plate thickness is again $L = \lambda/4 = 0.125 \times 10^{-6}\,m$

e) Assume a right-handed circularly polarized input
i.e. at $y=0$: $\vec{E} = E_0(\hat{z} - j\hat{x})$ for simplicity assume $\underline{|E_0| = 1\,V/m}$

At $y \neq 0$: $\vec{E} = \hat{z} e^{-jk^e y} - j\hat{x} e^{-jk^o y} = e^{-jk^e y}(\hat{z} - j\hat{x}\, e^{+j(k^e - k^o)y})$

At the output we want $\vec{E} = e^{-jk^e L}(\hat{z} + j\hat{x})$ LEFT-HANDED

i.e. $(k^e - k^o)L = \pi \Rightarrow L = \dfrac{\pi}{k^e - k^o} \Rightarrow L = \dfrac{\lambda}{2(\sqrt{\varepsilon^e} - \sqrt{\varepsilon^o})} = \dfrac{\lambda}{2}$

$\Rightarrow \underline{L = 0.25 \times 10^{-6}\,m}$

f) The only way to rotate a linear polarization is to change the sign of one component relative to the other, i.e. if the input at $y=0$ is $\vec{E} = \hat{z} + A\hat{x}$ then the output at $y = L$ must be $\vec{E} = e^{-jk^e L}(\hat{z} - A\hat{x})$ where $\underline{A = \text{real}}$

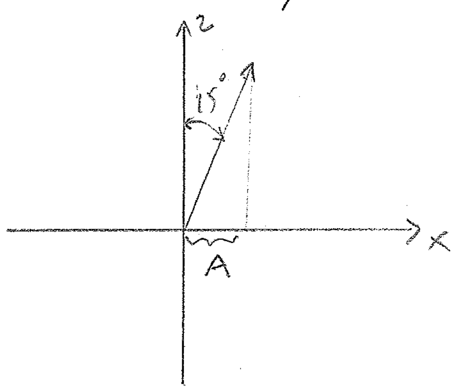
at any y $\quad \vec{E} = e^{-jk^e y}(\hat{z} + \hat{x} A e^{+j(k^o - k^e)y})$ $\quad$ (we have assumed again amplitude $= 1\,V/m$)

so we need $(k^o - k^e)L = \pi \Rightarrow L = \dfrac{\pi}{k^o - k^e} = \dfrac{\lambda}{2}$

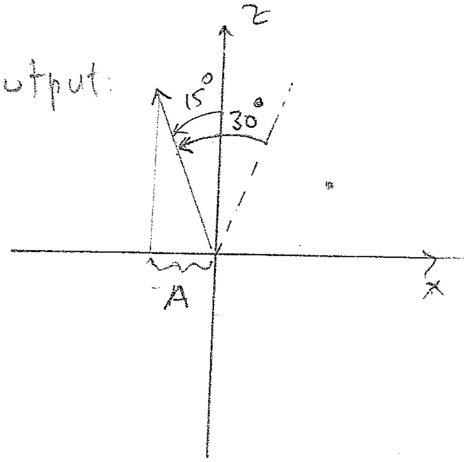
P. 6.4 f) cont'd

A must be chosen so that the angle of the input linear polarization w.r.t. to the z-axis is $\frac{\theta}{2} = \operatorname{atan}(A)$ where θ is the angle of polarization rotation. (we measure angle in the counter-clockwise direction)

So for $\theta = 30°$: $A = \tan(15°) \cong 0.268$ and the input polarization is:



and the output:



P. 6.5

$$\frac{\partial^2 \vec{d}(\vec{r},t)}{\partial t^2} + \frac{1}{\tau_c}\frac{\partial \vec{d}(\vec{r},t)}{\partial t} = -\frac{e}{m}\vec{E}(\vec{r},t)$$

a) Using phasors $\vec{d}(\vec{r})$ and $\vec{E}(\vec{r})$ $\frac{\partial}{\partial t} \to j\omega$ so (1) becomes:

$$-\omega^2 \vec{d}(\vec{r}) + j\frac{\omega}{\tau_c}\vec{d}(\vec{r}) = -\frac{e}{m}\vec{E}(\vec{r}) \Rightarrow \vec{d}(\vec{r}) = \frac{e}{m\omega^2}\frac{1}{\left(1-\frac{j}{\omega\tau_c}\right)}\vec{E}(\vec{r})$$

b) $\vec{P}(\vec{r}) = -Ne\vec{d}(\vec{r}) = -\frac{Ne^2}{m\omega^2}\frac{1}{1-j/\omega\tau_c}\vec{E}(\vec{r})$

c) $\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\left(1 - \frac{Ne^2}{m\varepsilon_0\omega^2}\frac{1}{1-j/\omega\tau_c}\right)\vec{E}(\vec{r}) = \varepsilon\vec{E}(\vec{r})$

P 6.5 c) cont'd

so $\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{Ne^2}{m\varepsilon_0\omega^2} \frac{1}{1 - j/\omega\tau_c}\right) =$

$$= \varepsilon_0 \left(1 - j \frac{\frac{Ne^2}{m}\tau_c}{\omega\varepsilon_0 (1 + j\omega\tau_c)}\right)$$

set $\boxed{\sigma = \frac{Ne^2\tau_c}{m}}$ so $\varepsilon(\omega) = \varepsilon_0\left[1 - j \frac{\sigma}{\omega\varepsilon_0 (1 + j\omega\tau_c)}\right]$

d) $\omega\tau_c << 1 \Rightarrow \varepsilon(\omega) \simeq \varepsilon_0 \left(1 - j\frac{\sigma}{\omega\varepsilon_0}\right) \Rightarrow \varepsilon(\omega)$ is that of a conductor.

e) $\omega\tau_c >> 1 \Rightarrow \varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\sigma}{\omega^2\varepsilon_0\tau_c}\right)$ plug expression for σ back in.

$$\Rightarrow \varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{Ne^2}{m\varepsilon_0\omega^2}\right)$$

$$\Rightarrow \varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$ expression for plasma $\varepsilon(\omega)$

GOLD:

f) $\tau_c = 10^{-13}$ sec $N = 1.5 \times 10^{28}\, m^{-3}$ $e = 1.6 \times 10^{-19}$ C $m = 9.1 \times 10^{-31}$ Kg

$\sigma = \frac{Ne^2\tau_c}{m} \simeq \underline{4.22 \times 10^7 \text{ S/m}}$

P 6.5 cont'd.

g) skin depth for conductor: $\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$

At $f = 1\,Hz$ $\delta = 7.75 \times 10^{-2}\,m$

At $f = 1\,KHz$ $\delta = 2.45 \times 10^{-3}\,m$

At $f = 1\,MHz$ $\delta = 7.75 \times 10^{-5}\,m$

At $f = 1\,GHz$ $\delta = 2.45 \times 10^{-6}\,m$

At $f = 1\,THz$ $\delta = 7.75 \times 10^{-8}\,m$

h) plasma frequeny of gold: $\omega_p = \sqrt{\frac{Ne^2}{m \varepsilon_0}} = 6.9 \times 10^{15}\,rad/sec$.

i) wavelength corresponding to ω_p is $\lambda = \frac{2\pi c}{\omega_p} \simeq 273\,nm$

which is in the UV range