ECE 303: Electromagnetic Fields and Waves

Fall 2007

Homework 6 **Due on Oct. 05, 2007 by 5:00 PM**

Reading Assignments:

i) Review the lecture notes.
ii) Review sections 1.5, 3.3-3.6 of the paperback book *Electromagnetic Waves*.

Special Note: Graders have been instructed to take off points (as much as 50%) if proper units are not included in your answers. You must specify the correct units with your numerical answers.

Problem 6.1: (Conductive Media)

Consider a plane wave in a conductive medium given by the expression:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-jk\,z}$$

The medium has a conductivity σ and a dielectric permittivity ε. As explained in the lectures, the wave decays because the wave looses energy due to dissipation in the medium. The Poynting theorem is:

$$-\nabla . \vec{S}(\vec{r},t) = \frac{\partial}{\partial t}\left[W_e(\vec{r},t) + W_m(\vec{r},t)\right] + \vec{J}(\vec{r},t).\vec{E}(\vec{r},t)$$

The time-average version of the Poynting theorem for time-harmonic fields is (as in homework 4):

$$-\left\langle \nabla . \vec{S}(\vec{r},t)\right\rangle = \left\langle \vec{J}(\vec{r},t).\vec{E}(\vec{r},t)\right\rangle$$

Using phasors this becomes:

$$-\frac{1}{2}\mathrm{Re}\left[\nabla . \vec{S}(\vec{r})\right] = \frac{1}{2}\mathrm{Re}\left[\vec{J}(\vec{r}).\vec{E}^*(\vec{r})\right]$$

This is saying that if a wave is slowly loosing energy, and so the Poynting vector is a function of position, then there must be dissipation going on. Verify the above relation for the plane wave propagating in a conductive medium given above (without making any "lossy dielectric" or "imperfect metal" approximations). You need to evaluate the left and right sides of the equation separately and show that they are equal.

Hint: First show (without making any approximations) that for any conductive medium: $2k'\,k'' = \omega\,\mu_o\,\sigma$.

Problem 6.2: (Ground Penetrating Radar - GPR)

A conductor is considered a good conductor for the frequency ω of interest if the loss tangent $\sigma/\omega\varepsilon$ is much greater than unity (i.e. $\sigma/\omega\varepsilon >> 1$) and a bad conductor if $\sigma/\omega\varepsilon << 1$. These conditions can also be stated in terms of the **dielectric relaxation time** $\tau_d = \varepsilon/\sigma$:

Good conductor: $\omega\tau_d << 1$ (This is also the "imperfect metal" case discussed in the lecture notes)

Bad conductor: $\omega\tau_d >> 1$ (This is also the "lossy dielectric" case discussed in the lecture notes)

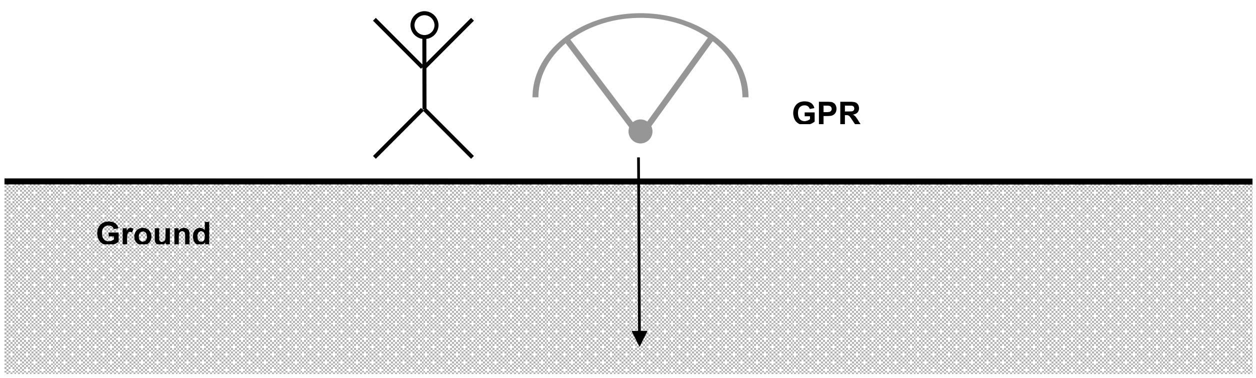
Solid ground has a conductivity of approximately 5×10^{-3} S/m and a permittivity ε equal to $10\varepsilon_o$. You need to design a ground penetrating radar (GPR). You have at our disposal two sources of electromagnetic radiation – one at a frequency of 10 kHz and the other one at a frequency of 100 MHz.

a) Figure out if ground is a good or a bad conductor at each of the two frequencies: 10 kHz and 100 MHz (don't forget that $\omega = 2\pi f$).

b) Figure out the penetration depth of electromagnetic radiation in ground at the two frequencies: 10 kHz and 100 MHz. Take penetration depth to be the depth at which the **time average power** in the electromagnetic radiation going into the ground drops to 1% of its value at the surface.

c) To clarify things a bit, calculate and plot (using matlab or your favorite plotting software) the exact penetration depth (as defined in part (b) above), without making any "good conductor" or "bad conductor" approximations, as a function of the frequency from 1 KHz to 1 GHz using a log frequency scale. Indicate in your plot frequency regions for which the ground acts like a "good conductor" and for which the ground acts like a "lossy dielectric" (or a "bad conductor").

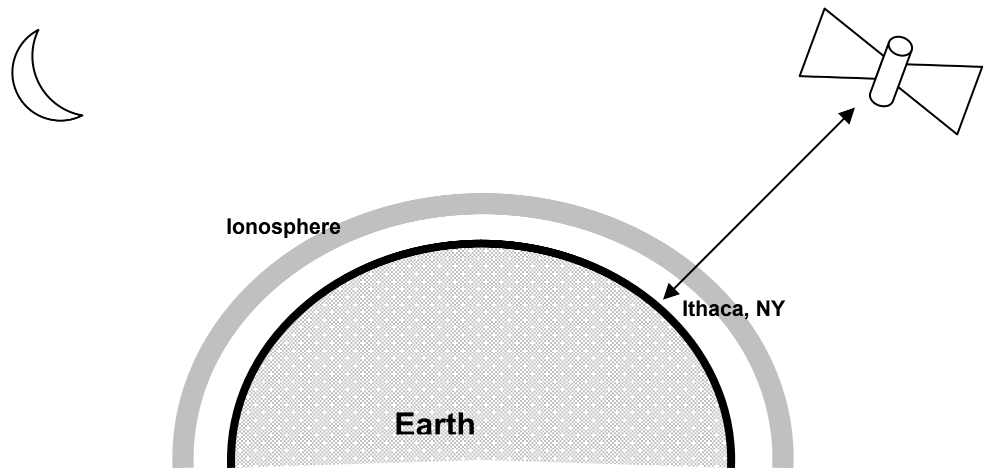
d) If you have to use the radar to image an object at a depth of 50 m, and you need the power reflected back to you from the object to have dropped to no more than 1% of its original starting value, what frequency (in Hz and not in rad/s) would you choose for the radar in order to image the smallest possible object. **Hint:** It is difficult to image objects smaller than the wavelength of the radiation being used for imaging.



Visit this link if you want to see some nice pictures from GPRs: http://www.geomodel.com/

Problem 6.3: (Atmospheric plasmas)

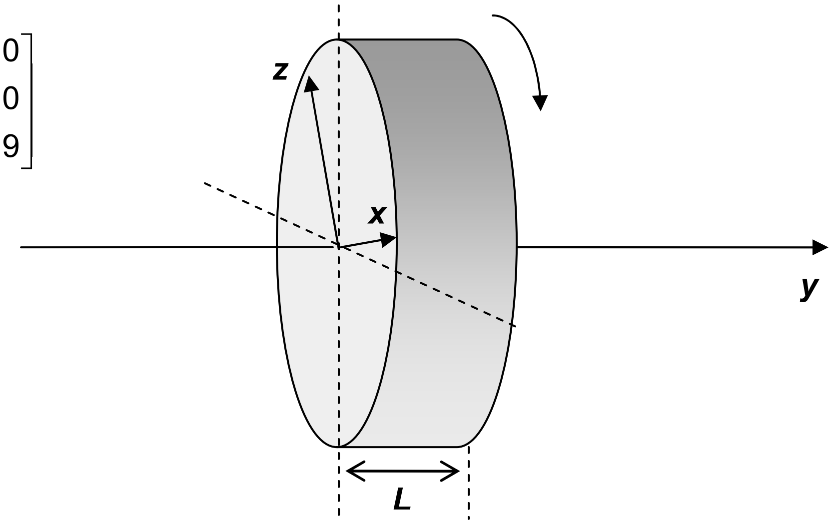
The Earth's **ionosphere** can be modeled as a Plasma. The gas molecules in the upper layers of the Earth's atmosphere are ionized by Solar and Cosmic radiation resulting in a Plasma. Suppose the density N of (singly ionized) molecules in the ionosphere is 10^{12} 1/m^3. The electron mass m is 9.1×10^{-31} kg. The electron charge e is $1.6x10^{-19}$ Coulombs. You have been hired by NASA to design a wireless communication system to communicate with a deep space probe. You need to select a frequency for your communication system (i.e. the frequency of the electromagnetic wave with which you will communicate with the deep space probe). What is the smallest possible frequency (in Hz NOT in rad/sec) that you can select and still be able to communicate with the deep space probe from somewhere on Earth without having your signals reflect back from the **ionosphere**.



Problem 6.4: (Uniaxial medium and waveplates)

Consider a uniaxial medium with the permittivity tensor given by:

$$\overline{\overline{\varepsilon}} = \varepsilon_o \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$



Waveplates are used in optics to control the polarization of light. In this problem, you will use the uniaxial medium to design waveplates for operation with green light (wavelength of green light in free-space is equal to 0.5×10^{-6} m).

For the following parts, assume that the waveplate can be rotated around the *y*-axis to suitably and optimally orient the principal axes of the waveplate w.r.t. the polarization of the incident wave as desired by the application. It is critical that you understand what is being said here,

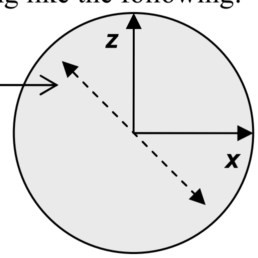
The diagram above shows the principal axes of the waveplate (the *y*-*z*-*x* directions). **All waves will be assumed to be traveling in the +*y*-direction in the medium**. The answers to questions below are not trivial and will test your understanding of the material.

a) Which axis is the extra-ordinary axis of the medium? And which axes are the ordinary axes?

b) Which axes are the slow axes of the medium? And which axes are the fast axes?

c) What is the minimum thickness L of the waveplate (answer in meters) that will let one convert any linearly polarized incident wave into a right-hand circularly polarized output wave? Explain in detail your answer. How would you orient the axis of the waveplate w.r.t. the incident polarization direction to get a right-hand circularly polarized output wave for the minimum thickness you calculated?

Hint: Your answer should include a diagram that shows the orientation of the principal axes of the waveplate w.r.t. the polarization of the incident wave as desired by the application. For example, to answer part (c) you should draw something like the following:



and say that "*to get right-hand circular polarization I will rotate the waveplate so that the direction of incident linear polarization is at 45-degrees to the z-axis of the waveplate (as shown in the figure) and then choose thickness* L *of the waveplate such that ………*". You should use the same procedure to answer all the parts that follow.

d) What is the minimum thickness L of the waveplate (answer in meters) that will let one convert any linearly polarized incident wave into a left-hand circularly polarized output wave? Explain in detail your answer. How would you orient the axis of the waveplate w.r.t. the incident polarization direction to get a left-hand circularly polarized output wave for the minimum thickness you calculated?

e) What is the minimum thickness L of the waveplate (answer in meters) that will let one convert a right-hand circularly polarized incident wave into a left-hand circularly polarized output wave and vice-versa? Explain in detail your answer.

f) What is the minimum thickness L of the waveplate (answer in meters) that will let one rotate the polarization direction of any linearly polarized incident wave through an arbitrary angle at the output? Explain in detail your answer. How would you orient the axis of the waveplate w.r.t. the incident polarization direction to get the desired rotation angle for the output wave for the minimum thickness you calculated supposing the desired rotation angle was 30-degrees?

Problem 6.5: (Real metals: conductors or plasmas?)

In this course you were told that the response of metals to an electromagnetic wave can be modeled as that of conductors with conductivity σ, and can be described by an effective dielectric permittivity given by:

$$\varepsilon_{eff}(\omega) = \varepsilon_o \left(1 - j\frac{\sigma}{\omega \varepsilon_o}\right)$$

Notice the switch from ε to ε_o in the above expression compared to lecture notes for simplicity. You were also told that electrons (and atoms) in metals can be modeled as a plasma with a permittivity given by:

$$\varepsilon(\omega) = \varepsilon_o \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

And so the question arises which one of the above expressions gives the correct permittivity of real metals. In this problem you will explore this question more carefully. It turns out that both the above expressions can be correct depending on the frequency of the electromagnetic wave.

Consider a metal in which the density of electrons is N. The electron mass is m. The electron charge is e. In the presence of an electromagnetic wave, the electrons are accelerated by the E-field. However, in all real metals as a result of electron collisions with impurities or defects in the metal the average motion of the electrons is not really described by the simple equation in the lecture notes:

$$\frac{\partial^2 \vec{d}(\vec{r},t)}{\partial t^2} = -\frac{e}{m} E(\vec{r},t)$$

but by the equation:

$$\frac{\partial^2 \vec{d}(\vec{r},t)}{\partial t^2} + \frac{1}{\tau_c}\frac{\partial \vec{d}(\vec{r},t)}{\partial t} = -\frac{e}{m} E(\vec{r},t) \quad \text{———} \quad (1)$$

where the constant τ_c is the mean time between collisions, and $1/\tau_c$ is the collision frequency. The effect of collisions on electron motion has been included as a damping effect in equation (1). Collisions damp the motion of electrons in the same way as friction damps the motion of any real pendulum.

a) Assuming time-harmonic fields, convert equation (1) into phasor notation and solve it to find the value of the phasor $\vec{d}(\vec{r})$ in terms of the E-field phasor $\vec{E}(\vec{r})$.

b) Following the recipe outlined in the lecture notes, find an expression for the polarization phasor $\vec{P}(\vec{r})$ (capital "P").

c) Using your results from part (b), find an expression for the dielectric permittivity $\varepsilon(\omega)$ of real metals, and show that it can be written in the form:

$$\varepsilon(\omega) = \varepsilon_o\left(1 - j\frac{\sigma}{\omega\varepsilon_o\,(1 + j\,\omega\,\tau_c)}\right)$$

What is the expression for the conductivity σ that comes out of your analysis? Those of you taking ECE315, does the expression for conductivity seem correct?

d) When the frequency of electromagnetic wave is much less than the electron collision frequency (i.e. when $\omega\,\tau_c << 1$) show that the expression for permittivity in (c) reduces to that of a conductor.

e) When the frequency of electromagnetic wave is much larger than the electron collision frequency (i.e. when $\omega\,\tau_c >> 1$) show that the expression for permittivity in (c) reduces to that of a plasma.

Gold characteristics:

Gold is one of the most commonly used materials in microwave devices and circuits, integrated antennas, and in metallic mirrors for optics. Below you will look at some of the features of this material.

f) The mean time τ_c between collisions in gold is around 10^{-13} sec. So for frequencies much much smaller than 1.6 THz (note that 1.0 THz $= 10^{12}$ Hz), gold can be approximately modeled as a conductor. Find the conductivity σ of gold assuming:

$N = 1.5 \times 10^{28}\ 1/\mathrm{m}^3$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$m = 9.1 \times 10^{-31}\ \mathrm{kg}$

$\tau_c = 10^{-13}\ \mathrm{Sec}$

g) Using your answer from part (f), find the skin-depth in gold of electromagnetic waves of frequencies 1 Hz, 1 KHz, 1 MHz, and 1 GHz, and 1 THz.

h) For frequencies much much larger than 1.6 THz, gold can be approximately modeled as a plasma. Find the plasma frequency of gold assuming:

$N = 1.5 \times 10^{28}\ 1/\mathrm{m}^3$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$m = 9.1 \times 10^{-31}\ \mathrm{kg}$

$\tau_c = 10^{-13}\ \mathrm{Sec}$

i) Figure out (from a Google search perhaps) what "color" of light (e.g. X-ray, Ultra-Violet, Violet, Blue, Green, Red, Infra-Red, etc) does the plasma frequency you calculated above corresponds to.