ECE 303 HOMEWORK #4 Solutions

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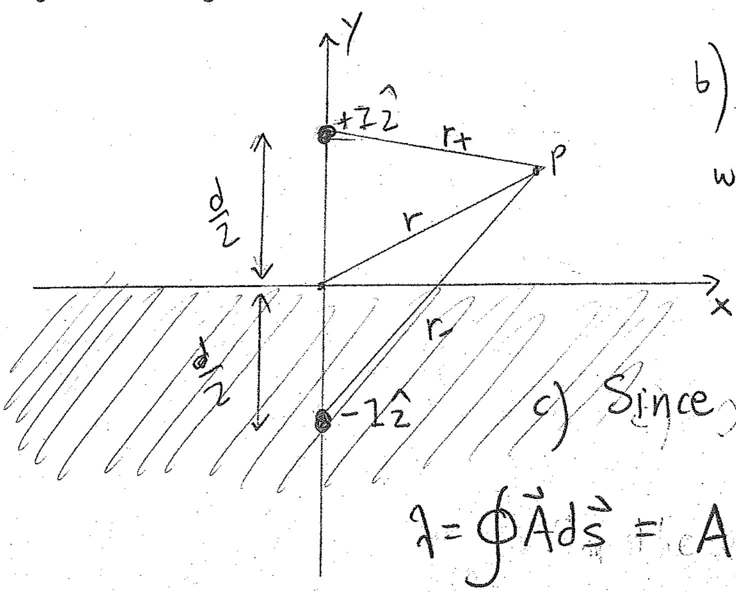
4.1

a) Image current: $-I\hat{z}$



b) $\vec{A} = A_z\hat{z} = \hat{z}\,\frac{\mu_0 I}{2\pi}\ln\left(\frac{r_-}{r_+}\right)$

where:

$$r_\pm = \left[x^2 + \left(y \mp \frac{d}{2}\right)^2\right]^{1/2}$$

c) Since $\vec{A}$ has only z-component:

$$\lambda = \oint \vec{A}\,d\vec{s} = A_z(r_- = d, r_+ = a) - A_z(r_- = d, r_+ = d)$$

$$= \frac{\mu_0 I}{2\pi}\ln\left(\frac{d}{a}\right) - 0 \Rightarrow \lambda = \frac{\mu_0 I}{2\pi}\ln\left(\frac{d}{a}\right)$$

d) $\lambda = L\cdot I \Rightarrow L = \frac{\mu_0}{2\pi}\ln\left(\frac{d}{a}\right)$

e) We need the tangential component of the magnetic field on the metal surface:

$$H_x\Big|_{y=0} = \frac{1}{\mu_0}\frac{\partial A_z}{\partial y}\Big|_{y=0} = \frac{I}{\pi}\,\frac{d/2}{x^2+\left(\frac{d}{2}\right)^2}$$

$$\vec{K} = \hat{y}\times\hat{x}\,H_x \Rightarrow \vec{K} = -\hat{z}\,\frac{I}{\pi}\,\frac{d/2}{x^2+\left(\frac{d}{2}\right)^2}$$

$$\vec{I} = \int_{-\infty}^{\infty}\vec{K}(x)\,dx = -I\hat{z}$$

4.2

a) $\vec{A} = A_z(r,\varphi)\hat{z}$ $\quad \vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} \Rightarrow \begin{cases} H_r = \frac{1}{\mu_0}\frac{1}{r}\frac{\partial A_z}{\partial\varphi} \\ H_\varphi = -\frac{1}{\mu_0}\frac{\partial A_z}{\partial r} \\ H_z = 0 \end{cases}$

b) $A_z^{in}(r,\varphi) = Ar\cos\phi$

$A_z^{out}(r,\varphi) = B\frac{1}{r}\cos\phi$

c) Continuity of perpendicular magnetic field component:

$$\left.\frac{1}{a}\frac{\partial A_z^{in}}{\partial\varphi}\right|_{r=a} = \left.\frac{1}{a}\frac{\partial A_z^{out}}{\partial\varphi}\right|_{r=a} \Rightarrow B = Aa^2 \quad (1)$$

Discontinuity of tangential magnetic field component:

$$\hat{r}\times\hat{\varphi}\left(-\left.\frac{\partial A_r^{out}}{\partial r}\right|_{r=a} + \left.\frac{\partial A^{in}}{\partial r}\right|_{r=a}\right) = K_0\cos\varphi\,\hat{z} \Rightarrow \frac{B}{a^2} + A = K_0 \quad (2)$$

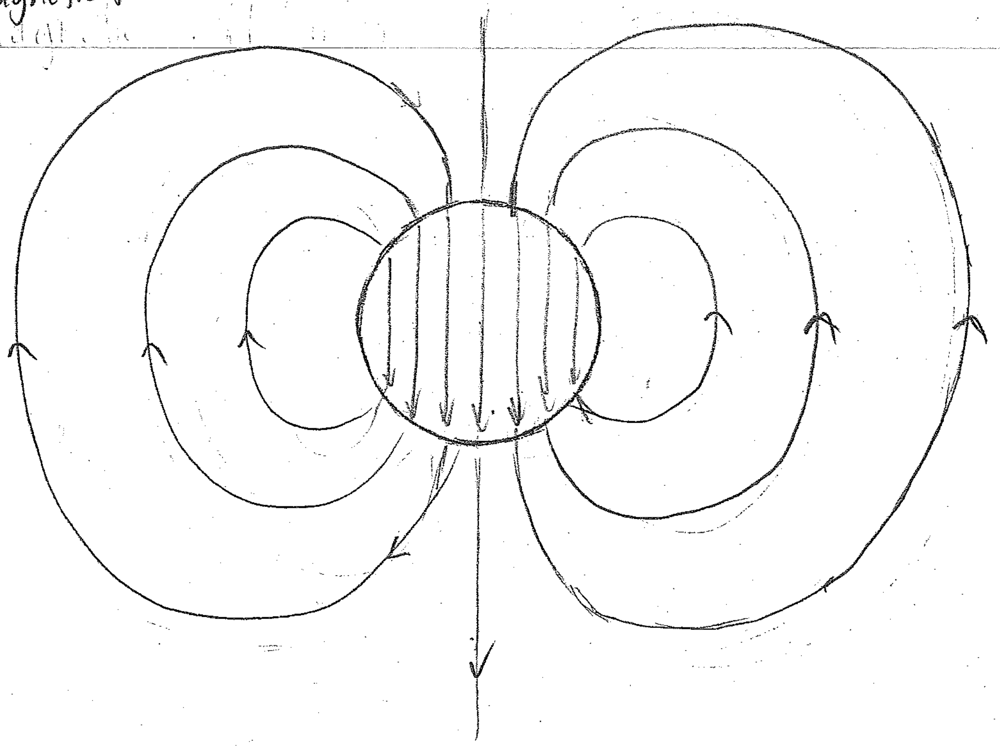
d) From (1), (2) $\quad A = \frac{K_0}{2} \quad B = K_0\frac{a^2}{2}$

e) $H_r^{in} = -\frac{K_0}{2}\sin\phi \qquad H_\varphi^{in} = -\frac{K_0}{2}\cos\phi$

$H_r^{out} = -\frac{K_0}{2}\frac{a^2}{r^2}\sin\phi \qquad H_\varphi^{out} = \frac{K_0}{2}\frac{a^2}{r^2}\cos\phi$

4.2 cont'd

f) Magnetic field lines.



4.3

a) Faraday's Law: $\oint \vec{E}\cdot d\vec{s} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{S} = -\frac{d\Phi(t)}{dt} = -g$

Taking a contour that goes through the metal wire and the gap and taking into account that $\vec{E}=0$ inside the wire we have:

$$\oint \vec{E}\cdot d\vec{s} = E_\varphi a = -g \Rightarrow E_\varphi = -g/a \qquad \vec{E} = -\frac{g}{a}\hat{\varphi}$$

b) Taking the same contour, but now $\vec{E} \neq 0$ in the wire:

$$\oint \vec{E} d\vec{s} = E_\varphi 2\pi c = -g \Rightarrow E_\varphi = -\frac{g}{2\pi c} \qquad \vec{E} = -\frac{g}{2\pi c}\hat{\varphi}$$

$$\vec{J} = \sigma \vec{E} \Rightarrow \vec{J} = -\frac{g\sigma}{2\pi c}\hat{\varphi} \quad \text{and} \quad \vec{I} = \pi\left(\frac{b}{2}\right)^2 \vec{J} \Rightarrow \vec{I} = -\frac{b^2 g\sigma}{8c}\hat{\varphi}$$

4.4

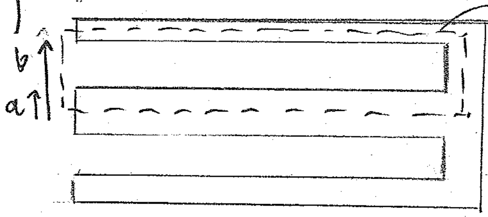
a) Ampere's Law: $\oint \vec{H} d\vec{s} = I \Rightarrow H_\varphi \cdot 2\pi r = I \Rightarrow$

$$\Rightarrow H_\varphi = \frac{I}{2\pi r} \Rightarrow \vec{H} = \frac{I}{2\pi r}\hat{\varphi}$$

b) $$\lambda = \iint \mu_0 \vec{H} \cdot d\vec{a} = \frac{\mu_0 w I}{2\pi}\int_a^b \frac{dr}{r} \Rightarrow \lambda = \frac{\mu_0 w I}{2\pi} \ln\left(\frac{b}{a}\right)$$

c) $$\lambda = L \cdot I \Rightarrow L = \frac{\mu_0 w}{2\pi}\ln\left(\frac{b}{a}\right)$$

d)



$$\oint \vec{E} d\vec{s} = -\frac{d\lambda}{dt} = -\frac{\mu_0 w}{2\pi}\ln\left(\frac{b}{a}\right)\frac{dI}{dt} = -L\frac{dI}{dt}$$

since $\vec{E} = 0$ inside the metal:

$$\oint \vec{E} d\vec{s} = -\int_a^b E_r(z=0)\,dr = -L\frac{dI}{dt}$$

$$\Rightarrow \int_a^b E_r(z=0)\,dr = L\frac{dI}{dt}$$

e) In the gap: $\vec{E} = E_r(r,z)\hat{r}$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} = -\frac{\mu_0}{2\pi r}\frac{dI}{dt}\hat{\varphi} \Rightarrow \frac{\partial E_r}{\partial z}\hat{\varphi} = -\frac{\mu_0}{2\pi r}\frac{dI}{dt}\hat{\varphi}$$

$$\Rightarrow \frac{\partial E_r}{\partial z} = -\frac{\mu_0}{2\pi r}\frac{dI}{dt}$$

Integrating from $z=z$ to $z=w$ both sides:

$$\int_z^w \frac{\partial E_r}{\partial z}dz = -\frac{\mu_0}{2\pi r}\frac{dI}{dt}\int_z^w dz = -\frac{\mu_0}{2\pi r}(w-z)\frac{dI}{dt}$$

4.4 cont'd

$$\Rightarrow E_r(r,w) - E_r(r,z) = -\frac{\mu_0}{2\pi r}(w-z)\frac{dI}{dt}$$

Tangential component on metal surface is zero: $E_r(r,w) = 0$

so: $E_r(r,z) = \frac{\mu_0}{2\pi r}(w-z)\frac{dI}{dt} \Rightarrow \vec{E} = \frac{\mu_0}{2\pi r}(w-z)\frac{dI}{dt}\hat{r}$

check: $\nabla\cdot\vec{E} = 0 = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = 0$

$$\int_a^b E_r(r,0)\,dr = \frac{\mu_0 w}{2\pi}\ln\left(\frac{b}{a}\right)\frac{dI}{dt} = L\frac{dI}{dt}$$

4.5

a) $\vec{E} = \frac{J_0}{\sigma}\sin(\omega t)\,\hat{z}$

b) $\vec{H} = H_\varphi\hat{\varphi}$ Ampere's Law: $H_\varphi \cdot 2\pi r = J_0\pi r^2\sin(\omega t)$

$$\Rightarrow H_\varphi = \frac{J_0 r}{2}\sin(\omega t)$$

c) $P_d = \left\langle \iiint \vec{J}\cdot\vec{E}\,dV \right\rangle = \left\langle \pi a^2 L\frac{J_0^2}{\sigma}\sin^2(\omega t)\right\rangle = \frac{1}{2}\frac{J_0^2}{\sigma}\pi a^2 L$.

d) $R = \frac{V}{I} = \frac{E_z L}{J_z\pi a^2} \Rightarrow R = \frac{L}{\sigma\pi a^2}$

$\langle I^2 R\rangle = \left\langle J_0^2\pi^2 a^4\cdot\frac{L}{\sigma\pi a^2}\sin^2(\omega t)\right\rangle = \frac{1}{2}\frac{J_0^2}{\sigma}\pi a^2 L$ same as c)

4.5 cont'd

e) $-\oiint (\vec{E}\times\vec{H})\cdot d\vec{a}$: $-$ $\vec{E}\times\vec{H} = \frac{J_0}{\sigma}\sin(\omega t)\,\hat{z}\times\frac{J_0 a}{2}\sin(\omega t)\,\hat{\varphi}$

$$= -\hat{r}\,\frac{J_0^2 a}{2\sigma}\sin^2(\omega t)$$

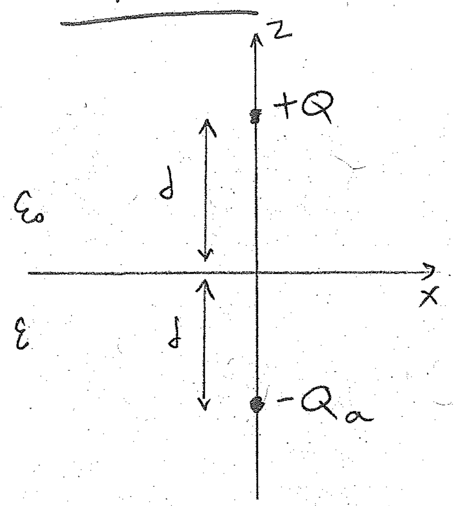
$$d\vec{a} = 2\pi a\,dz\cdot\hat{r}$$

So $-\oiint \vec{E}\times\vec{H}\cdot d\vec{a} = \frac{J_0^2 a}{2\sigma}\sin^2(\omega t)\,2\pi a L = \frac{J_0^2}{\sigma}\pi a^2 L\sin^2(\omega t)$

$$\left\langle -\oiint \vec{E}\times\vec{H}\cdot d\vec{a}\right\rangle = \frac{1}{2}\frac{J_0^2}{\sigma}\pi a^2 L.$$

f) e) and e) agree.

4.6



a) From Homework #3: $Q_a = \frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}Q$

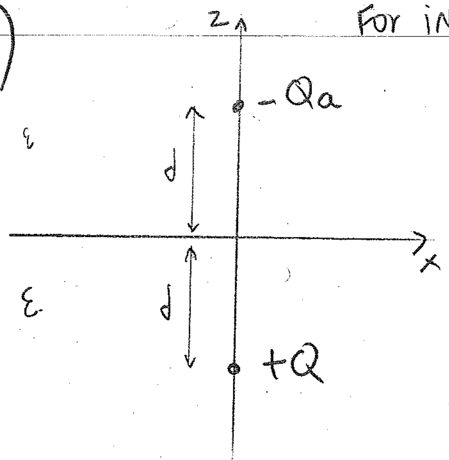
So Lorentz force on the charge $+Q$ is:

$$\vec{F} = -\frac{QQ_a}{4\pi\varepsilon_0(2d)^2} = -\frac{Q^2}{4\pi\varepsilon_0(2d)^2}\,\frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\,\hat{z}$$

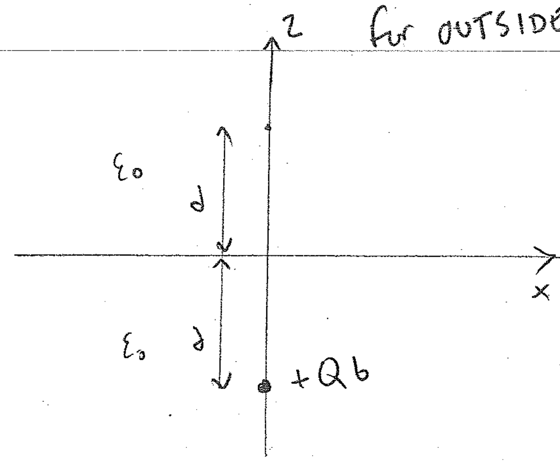
Attractive force so the charge $+Q$ is pulled towards the interface.

4.6 cont'd

b) For INSIDE



for OUTSIDE



$$\varphi_{in} = \frac{Q}{4\pi\varepsilon r_+} - \frac{Q_a}{4\pi\varepsilon r_-} \qquad \varphi_{out} = \frac{Q_b}{4\pi\varepsilon_0 r_+}$$

Boundary conditions: continuity of potential $\quad \varphi_{in}\Big|_{z=0} = \varphi_{out}\Big|_{z=0} \Rightarrow \dfrac{Q - Q_a}{\varepsilon} = \dfrac{Q_b}{\varepsilon_0}$ (1)

discontinuity of normal E-field $\quad \varepsilon \dfrac{d\varphi_{in}}{dz}\Big|_{z=0} = \varepsilon_0 \dfrac{d\varphi_{out}}{dz}\Big|_{z=0} \Rightarrow Q + Q_a = Q_b.$ (2)

From (1), (2) $\quad Q_a = -\dfrac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0} Q \qquad Q_b = \dfrac{2\varepsilon_0}{\varepsilon + \varepsilon_0} Q$

$\Downarrow$

Image charge $-Q_a$ is actually Positive

c) Force on charge $+Q$: $\quad \vec{F} = \dfrac{+Q\,Q_a}{4\pi\varepsilon (2d)^2}\hat{z} = -\dfrac{Q^2}{4\pi\varepsilon}\dfrac{1}{(2d)^2}\dfrac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0}\hat{z}$

i.e. $+Q$ is pushed away from the interface