ECE 303 Homework #3 Solutions

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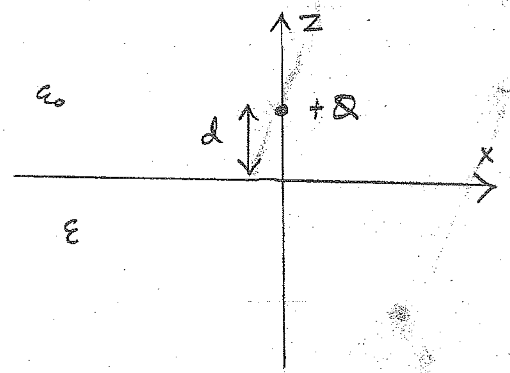
3.2

a) $\phi_{out}(\vec{r}) = \dfrac{Q}{4\pi\varepsilon_o r_+} - \dfrac{Q_a}{4\pi\varepsilon_o r_-}$ $\qquad \phi_{in}(\vec{r}) = \dfrac{Q_b}{4\pi\varepsilon r_+}$

b) The boundary conditions are:



i) $\phi_{out}(\vec{r})\Big|_{interface} = \phi_{in}(\vec{r})\Big|_{interface}$

ii) $\varepsilon_o \vec{E}_{out}\cdot\hat{z}\Big|_{interface} = \varepsilon\,\vec{E}_{in}\cdot\hat{z}\Big|_{interface}$

$\{$since $\sigma_u = 0\}$

c) Boundary condition (i) gives: $\dfrac{Q - Q_a}{\varepsilon_o} = \dfrac{Q_b}{\varepsilon}$

Boundary condition (ii) gives: $Q + Q_a = Q_b$

Solving these equations gives:
$$\begin{cases} Q_b = \dfrac{2}{1+\frac{\varepsilon_o}{\varepsilon}}\,Q \\[2ex] Q_a = \dfrac{1-\frac{\varepsilon_o}{\varepsilon}}{1+\frac{\varepsilon_o}{\varepsilon}}\,Q \end{cases}$$

c) if $\varepsilon = \varepsilon_o$ then $Q_b = Q$ and $Q_a = 0$.

d) when $\varepsilon \to \infty$, $Q_a \to Q$ i.e. the strength of the image charge (as seen from outside) is as if the dielectric were perfect metal.

3.4

a) $\phi_{out}(\vec{r}) = -E_o r\cos\theta + A\dfrac{\cos\theta}{r^2}$

$\phi_{in}(\vec{r}) = B r\cos\theta$

b) Boundary Conditions are:

(i) $\phi_{out}(\vec{r})\Big|_{r=a} = \phi_{out}(\vec{r})\Big|_{r=a}$

(ii) $\varepsilon_0 \vec{E}_{out} \cdot \hat{r}\Big|_{r=a} - \varepsilon_1 \vec{E}_{in} \cdot \hat{r}\Big|_{r=a} = \sigma_u = 0.$

c) (i) $\Rightarrow \quad -E_0 a \cos\theta + \frac{A\cos\theta}{a^2} = Ba\cos\theta \quad \Rightarrow -E_0 a^3 + A = Ba^3$

(ii) $\Rightarrow \quad \varepsilon_0 E_0 + 2\frac{A\varepsilon_0}{a^3} = -B\varepsilon_1 \Rightarrow \quad \varepsilon_0 E_0 a^3 + 2\varepsilon_0 A = -Ba^3\varepsilon_1.$

$\Rightarrow \quad A = \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} E_0 a^3 \qquad B = -\frac{3\varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} E_0$

d) Dipole-like term in $\phi_{out}(\vec{r})$ is $A\frac{\cos\theta}{r^2} = \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} E_0 a^3 \frac{\cos\theta}{r^2}$

Potential for a point-charge dipole oriented along z-axis is $\frac{qd}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} = \frac{|\vec{p}|}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$ where $\vec{p} = qd\hat{z}$

So for the dielectric sphere the induced dipole moment must be $= \vec{p} = 4\pi\varepsilon_0 \left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0}\right) E_0 a^3 \hat{z}$

e) $\vec{P} = \vec{p}N = 4\pi\varepsilon_0 \left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0}\right) E_0 a^3 N \hat{z}$

f) $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ and $\vec{E} = E_0\hat{z} \Rightarrow \chi_e = 4\pi\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0}\right) a^3 N$

g) $\varepsilon = \varepsilon_0(1 + \chi_e) = \varepsilon_0\left[1 + 4\pi\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0}\right) a^3 N\right]$

By choosing the value of the dot radius a, dot concentration N, and the dot permittivity ε_1, one can tailor the permittivity ε of the nano-dot medium to any desired value.

3.3

a) $\Phi_{out}(\vec{r}) = -E_0 r\cos\phi + \dfrac{A\cos\phi}{r}$ $\qquad \phi_{in}(\vec{r}) = 0$

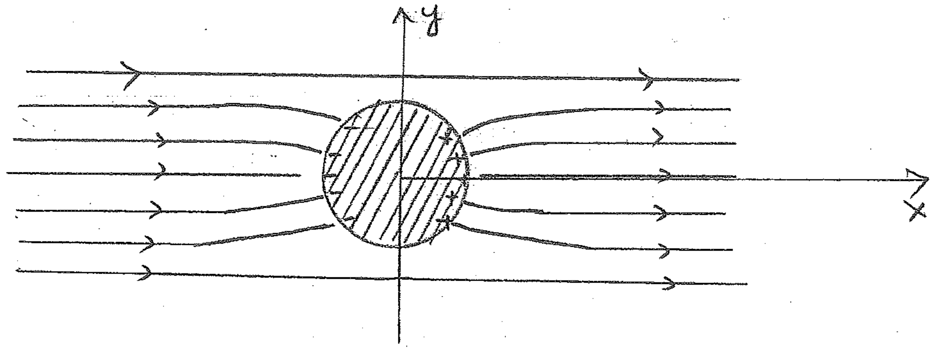
b) (i) $\Phi_{out}(\vec{r})\big|_{r=a} = \phi_{in}(\vec{r})\big|_{r=a} = 0$

c) $-E_0 a\cos\phi + A\dfrac{\cos\phi}{a} = 0 \Rightarrow A = E_0 a^2$

d) $\varepsilon_0 \vec{E}_{out}\cdot\hat{r}\big|_{r=a} = \sigma_u$

$\Rightarrow \sigma_u = \varepsilon_0 E_0 \cos\phi + \varepsilon_0 E_0 \cos\phi = 2\varepsilon_0 E_0 \cos\phi$

e)



3.4

a) $\Phi_1(r) = A\left(\frac{b}{r}\right) + B$ $\qquad \Phi_2(r) = C\left(\frac{b}{r}\right) + D$

b) $\Phi_1(r=a) = 0$ & $\Phi_2(r=c) = V$.

$-\varepsilon_1 \dfrac{\partial\Phi_1}{\partial r}\Big|_{r=b} = -\varepsilon_2 \dfrac{\partial\Phi_2}{\partial r}\Big|_{r=b}$ & $\Phi_1(r=b) = \Phi_2(r=b)$

$\Rightarrow$ $A\,\frac{b}{a} + B = 0$ $\qquad C\left(\frac{b}{c}\right) + D = V$

$A + B = C + D$.

$\varepsilon_1 \dfrac{A}{b} = \varepsilon_2 \dfrac{C}{b}$.

c). $A = \dfrac{-\varepsilon_2 V}{\varepsilon_2\left(\frac{b}{a} - 1\right) + \varepsilon_1\left(1 - \frac{b}{c}\right)}$ $\qquad B = \dfrac{\varepsilon_2\left(\frac{b}{a}\right)\cdot V}{\varepsilon_2\left(\frac{b}{a} - 1\right) + \varepsilon_1\left(1 - \frac{b}{c}\right)}$.

$$C = -\frac{\varepsilon_1 V}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)} \qquad D = \frac{V\left[\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\right]}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)}$$

d). $\varepsilon_0\left[\vec{E}_2\cdot\hat{r}\Big|_{r=b} - \vec{E}_1\cdot\hat{r}\Big|_{r=b}\right] = \sigma_p$

$$\sigma_p = \varepsilon_0\left[\frac{C}{b}-\frac{A}{b}\right] = \frac{\varepsilon_0}{b}C\left(1-\frac{\varepsilon_2}{\varepsilon_1}\right) = -\frac{\varepsilon_0(\varepsilon_1-\varepsilon_2)\frac{V}{b}}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)}$$

e) For the outer shell:

$$\sigma_u = -\varepsilon_2\vec{E}_2\cdot\hat{r}\Big|_{r=c} = -\varepsilon_2 C\frac{b}{c^2} = \frac{\varepsilon_1\varepsilon_2\left(\frac{b}{c^2}\right)V}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)}$$

For the inner sphere:

$$\sigma_u = \varepsilon_1\vec{E}_1\cdot\hat{r}\Big|_{r=a} = \varepsilon_1 A\frac{b}{a^2} = -\frac{\varepsilon_1\varepsilon_2\left(\frac{b}{a^2}\right)V}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)}$$

f) For the outer shell:

$$Q = 4\pi c^2\sigma_u = 4\pi\frac{\varepsilon_1\varepsilon_2 bV}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)}$$

For the inner sphere:

$$Q = 4\pi a^2\sigma_u = -4\pi\frac{\varepsilon_1\varepsilon_2 bV}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)}$$

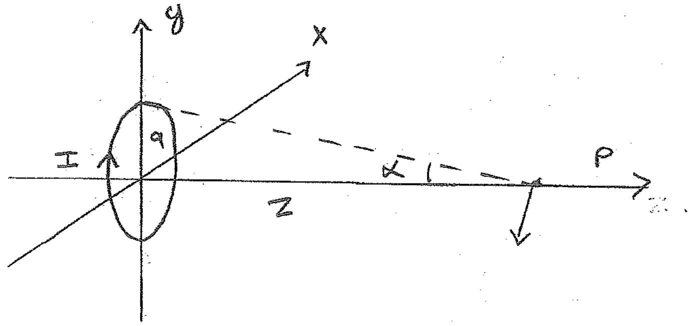
g) $$C = \frac{Q}{V} = 4\pi\frac{\varepsilon_1\varepsilon_2 b}{\varepsilon_2\left(\frac{b}{a}-1\right)+\varepsilon_1\left(1-\frac{b}{c}\right)}$$

3.6

a) By Symmetry, $H_x = 0$ at P.

b) By Symmetry, $H_y = 0$ at P.

c) Use Biot-Savart law $\quad H_z = \frac{I}{4\pi} \int \frac{d\vec{s}' \times \hat{n}_{\vec{s}' \to \vec{r}}}{|\vec{r} - \vec{s}'|^2}$



$|\vec{r} - \vec{s}'|^2 = a^2 + z^2$ for all $\vec{s}'$

The z-Components of magnetic field produced by all Current elements $d\vec{s}'$ add at point P. The z-component is obtained by multiplying by $\sin\alpha$.

$\sin\alpha = \frac{a}{\sqrt{a^2+z^2}} \Rightarrow H_z = -\frac{I}{4\pi} \frac{2\pi a}{(a^2+z^2)} \cdot \frac{a}{\sqrt{a^2+z^2}} = -\frac{I}{2\pi} \frac{(\pi a^2)}{(a^2+z^2)^{3/2}}$

3.5

a) Recall that $\rho_P = -\nabla \cdot \vec{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$. In the problem $\vec{P} = P_0 \hat{z}$, and so the z-component of $\vec{P}$ changes in the z-direction only at the top and bottom surfaces but not at the curved surface. At the upper surface $\sigma_P = +P_0$

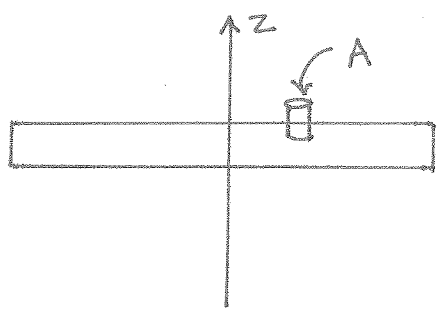
b) At the lower surface $\sigma_P = -P_0$

c) At the curved surface of the disc $\sigma_P = 0$

Another, and perphaps easier, way to figure out surface charge densities is to realize that:

$$\rho_p = -\nabla \cdot \vec{P} \quad \Rightarrow \quad \oint\!\!\oint \vec{P} \cdot d\vec{s} = -\iiint \rho_p \, dV$$

The above follows from the divergence theorem. It says that the total polarization vector flux coming out of a closed surface equals the -ve of the total polarization (or paired) charge enclosed. To find the surface charge density at the top surface I can draw the following closed surface in the form of a cylinder of area A:



$$\oint\!\!\oint \vec{P} \cdot d\vec{s} = -P_0 A$$

$$-\iiint \rho_p \, dV = -\sigma_p A$$

$$\Rightarrow \quad -P_0 A = -\sigma_p A$$

$$\Rightarrow \quad \sigma_p = +P_0 .$$

Similarly, the surface charge densities at the other surfaces can be found by drawing appropriate closed surfaces.

d) A +ve charge density at the top surface and a negative charge density at the lower surface implies an electric field equal to $\vec{E} = \frac{|\sigma_p|}{\varepsilon_0}(-\hat{z}) = -\frac{P_0}{\varepsilon_0}\hat{z}$ in the disc.

e) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(-\frac{P_0}{\varepsilon_0}\hat{z}\right) + P_0 \hat{z} = 0 .$