ECE 303: Electromagnetic Fields and Waves

Fall 2007

Homework 3 **Due on Sep. 14, 2007 by 5:00 PM**

Reading Assignments:

i) Review the lecture notes.

ii) Relevant sections of the online *Haus and Melcher* book for this week are 5.0-5.3, 5.9, 6.0-6.2. Note that the book contains more material than you are responsible for in this course. Determine relevance by what is covered in the lectures and the recitations. The book is meant for those of you who are looking for more depth and details.

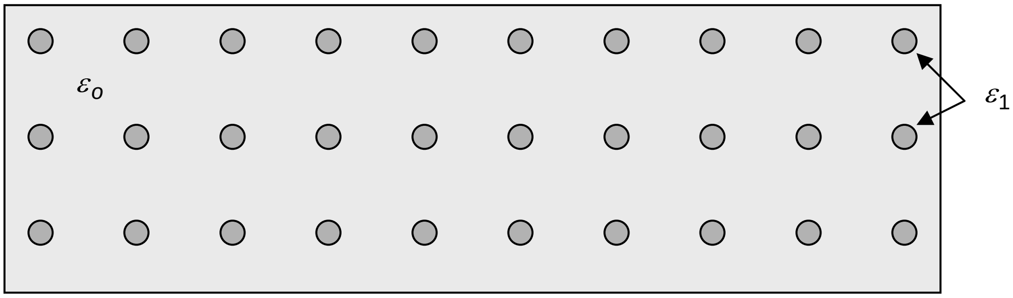
ii) This homework is long – start early.

Table of Solutions of Laplace's Equation

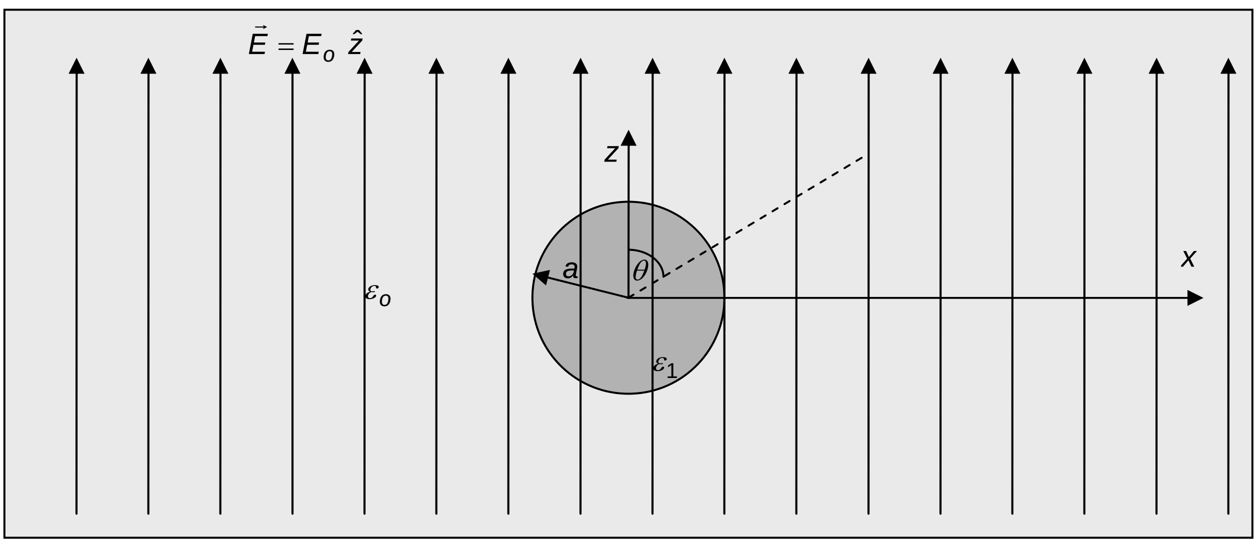
Spherical Coordinate System	Cylindrical Coordinate System
$\phi(\vec{r}) = A$ Constant potential	$\phi(\vec{r}) = A$ Constant potential
$\phi(\vec{r}) = \frac{A}{r}$ Spherically symmetric potential	$\phi(\vec{r}) = A\ln(r)$ Cylindrically symmetric potential
$\phi(\vec{r}) = A\,r\cos(\theta)$ Potential for uniform z-directed E-Field	$\phi(\vec{r}) = A\,r\cos(\phi)$ Potential for uniform x-directed E-Field
	$\phi(\vec{r}) = A\,r\sin(\phi)$ Potential for uniform y-directed E-Field
$\phi(\vec{r}) = A\frac{\cos(\theta)}{r^2}$ Potential for point-charge-dipole-like solution oriented along the z-axis	$\phi(\vec{r}) = A\frac{\cos(\phi)}{r}$ Potential for line-charge-dipole-like solution oriented along the x-axis
	$\phi(\vec{r}) = A\frac{\sin(\phi)}{r}$ Potential for line-charge-dipole-like solution oriented along the y-axis

Problem 3.1: (A nano-structured dielectric medium)

These days nano-technology is being used to "design" materials (as opposed to relying on nature) that have some desired characteristics. In this problem you will explore one such material made out of "nano-dots". Consider a material made up of nano-sized dielectric spheres (or nano-dots) of dielectric permittivity ε_1 embedded in another material that has the same permittivity as that of free-space (ε_o), as shown below.



We will consider this problem in different steps. Consider first a single dielectric sphere of radius a and of dielectric permittivity ε_1 embedded in a medium of permittivity ε_o , as shown below. A constant and uniform E-field has been applied in the +z-direction from outside.



a) Find trial solutions, $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$, for the potentials inside and outside the dielectric sphere. (Hint: $\phi_{out}(\vec{r})$ must have a term that has the same form as a dipole potential).

b) Write down all the boundary conditions (at least as many as the number of unknown constants in your answer to part (a) above) relevant to solving for the potentials, $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$.

c) Find all the unknown constants in your solutions in part (a) above by using all the boundary conditions in part (b) above.

d) Compare the dipole-like term in your solution $\phi_{out}(\vec{r})$ to that of a point-charge dipole potential (see your homework problem 2.1 solutions) and from that comparison figure out the dipole-moment $\vec{p}$ (small "p") of the polarized dielectric sphere. Make sure you get the correct units. (Hint: the dipole moment must be proportional to the applied E-field magnitude). This dipole moment has been "induced" in the dielectric sphere due to the external E-field.

Now come back to the medium made up of nano-sized dielectric spheres of dielectric permittivity ε_1 embedded in another medium of permittivity ε_o, as shown in an earlier picture. Suppose the dots are spaced reasonably far apart and so the field from one dot does not interact with the field of the other dots. Suppose that there are approximately N dots per unit volume.

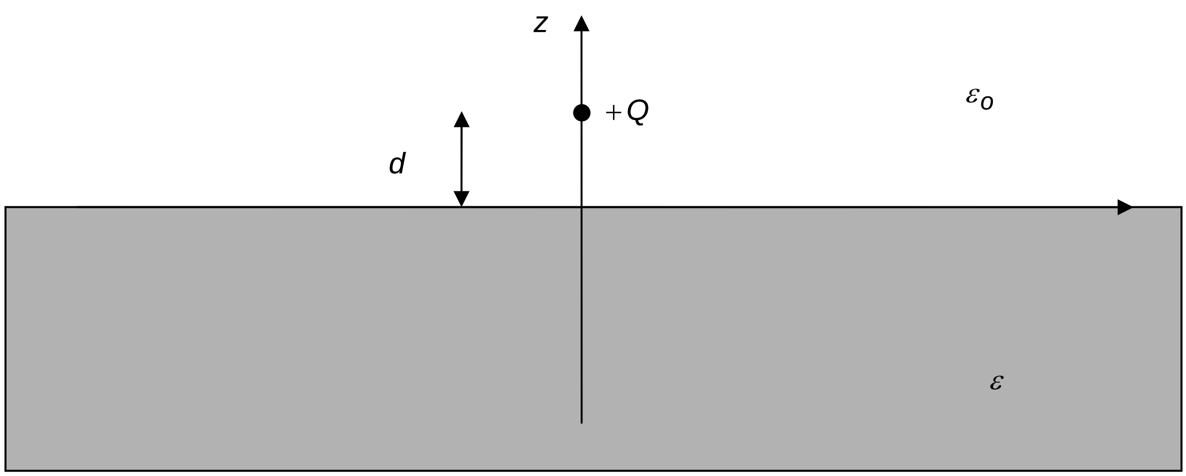
e) What is the polarization vector $\vec{P}$ (capital "P") of the nano-dot medium given that you know the dipole-moment of each dot? (Hint: the polarization vector must be proportional to the applied E-field magnitude).

f) From your answer to part (e), find the electrical susceptibility χ_e of the nano-dot medium.

g) From your answer in part (f), find the dielectric permittivity ε of the nano-dot medium.

Problem 3.2: (Dielectric image charges)

Consider a point charge $+Q$ sitting in free space at a distance d above a dielectric medium of permittivity ε, as shown below.



The electric field from the charge will get partially screened by the surface polarization charge density (paired surface charge density) that will exist at the surface of the dielectric medium. But unlike the perfect metal case, the electric field will not get fully screened out of the dielectric material.

In order to solve this problem, one needs to realize that the actual field solution, both inside and outside the dielectric medium, must be a superposition of the field due to the point charge and the field due to the surface polarization charge density (i.e. the paired charge density at the surface of the dielectric medium). A priori, we don't know what this surface charge density looks like so we will try to construct a guess solution.

We will assume that OUTSIDE the dielectric, the potential looks like the superposition of the potential of the original charge $+Q$ and the potential due to an image charge of strength $-Q_a$ sitting a distance d below the dielectric interface and that the whole space is filled with free space. The image charge has been assumed to have a different strength then the original charge because dielectric screening, unlike perfect metal screening, is not expected to be perfect.

For the potential INSIDE the dielectric we will assume that it looks like that of a charge of strength $+Q_b$ sitting outside the dielectric at a distance d away from the interface and that the whole space is filled with material of permittivity ε. This is because the actual field from the charge $+Q$ will get partially screened by the polarization (or paired) surface charged density at the surface of the dielectric.

a) Write an expression for the guess potential $\phi_{out}(\vec{r})$ outside the dielectric in terms of the distances r_+ and r_- from the charges $+Q$ and $-Q_a$, respectively, and for the guess potential $\phi_{in}(\vec{r})$ inside the dielectric in terms of the distance r_+ from the charge $+Q_b$.

b) You have two unknowns in your solution (the strength of the charges $-Q_a$ and $+Q_b$) and you need two boundary conditions. What are these two boundary conditions?

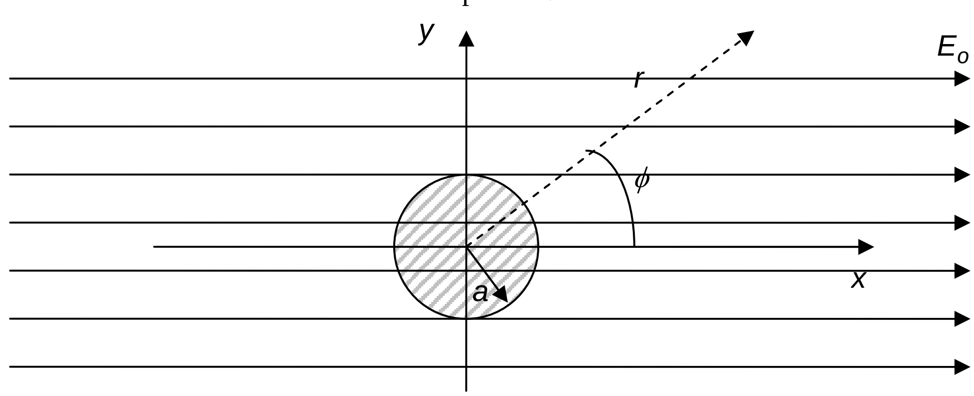
c) Using the boundary conditions from part (b) find the strength of the charges $-Q_a$ and $+Q_b$ in terms of the charge strength $+Q$ and the permittivities ε and ε_o.

d) Show from your result in part (c) that if $\varepsilon = \varepsilon_o$ then $Q_a = 0$ and $Q_b = Q$ which is what one would expect on physical grounds.

e) Show from your result in part (c) that when $\varepsilon \to \infty$ then the potential OUTSIDE looks as if the dielectric material were perfect metal.

Problem 3.3: (A perfect metal cylinder in a uniform electric field)

Consider an infinitely long (in the z-direction) perfect metal rod of radius a placed in a uniform and constant electric field pointing in the +x-direction as shown below. The figure below shows only the applied E-field lines – as if the metal rod were not present.



a) Find trial solutions, $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$, for the potentials inside and outside the metal rod. (Hint: $\phi_{out}(\vec{r})$ must have a term that has the same form as a line-charge dipole potential).

b) Write down all the boundary conditions (at least as many as the number of unknown constants in your answer to part (a) above) relevant to solving for the potentials, $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$.

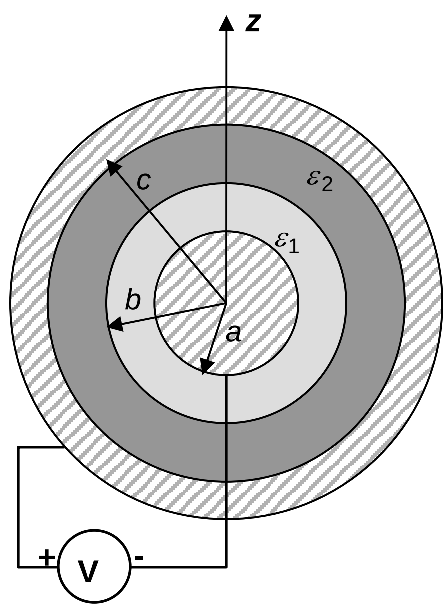
c) Find all the unknown constants in your solutions in part (a) above by using all the boundary conditions in part (b) above.

d) Find the surface charge density on the metal rod as a function of the angle ϕ.

e) Sketch the total E-field lines (note the figure above shows only the applied E-field lines – as if the metal rod were not present).

Problem 3.4: (A concentric spherical dielectric capacitor)

Consider a perfect metal sphere surrounded by a perfect metal spherical shell and connected to a voltage source as shown below. Completely ignore the physical presence of the voltage source and the connecting wires other than the fact that they establish a fixed potential difference. The space between the inner and outer spheres consists of two different dielectric layers as shown in the picture below.



a) Write trial solutions for the potentials, $\phi_1(\vec{r})$ and $\phi_2(\vec{r})$, in the two dielectric regions $a \le r \le b$ and $b \le r \le c$, respectively.

b) Write down all the boundary conditions (at least as many as the number of unknown constants in your answer to part (a) above) relevant to solving for the potentials, $\phi_1(\vec{r})$ and $\phi_2(\vec{r})$.

c) Find all the unknown constants in your solutions in part (a) above by using all the boundary conditions in part (b) above.

d) Find the sheet charge density (sign and magnitude) due to the **paired charges** at the interface between the two dielectrics.

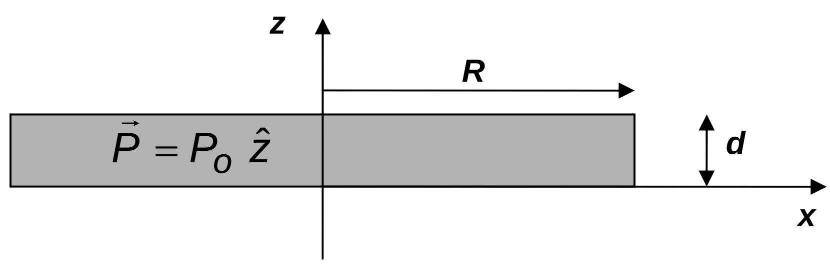
e) Find the surface charge densities (sign and magnitude) on the inner surface of the outer spherical metal shell and on the outer surface of the inner metal sphere.

f) Find the total charge (sign and magnitude) on the inner surface of the outer metal shell and also on the outer surface of the inner metal sphere.

g) Find the capacitance C (units: Farads) between the inner and outer spheres by taking the ratio of the total charge (found in part (e) above) and the applied voltage V .

Problem 3.5: (Ferroelectrics)

Ferroelectrics (as opposed to dielectrics) are materials that have their atoms/molecules all polarized in the same direction even when no external electric field is present. That is, a ferroelectric material has a built-in non-zero fixed polarization vector $\vec{P}$ that is independent of any external field. Some important semiconductors like Gallium Nitride (which is used these days in almost all the high power RF transmitters at base stations for mobile/wireless systems) are ferroelectric. In this problem you will explore the consequences of such a built-in polarization. Consider a circular disc of a ferroelectric material of thickness d that is much smaller than the radius R, as shown in the figure. The built-in polarization vector is given by $\vec{P} = P_o \ \hat{z}$.



a) Find the surface charge density due to the paired charges at the upper flat surface of the disc.

b) Find the surface charge density due to the paired charges at the lower flat surface of the disc.

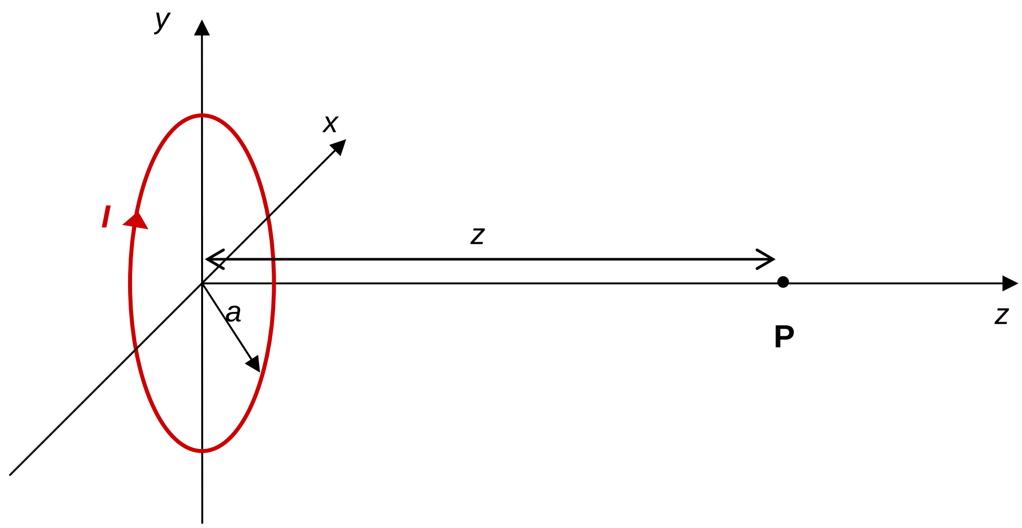
c) Find the surface charge density due to the paired charges at the curved outer surface of the disc.

d) Find the electric field (magnitude and direction) inside the ferroelectric disc. Hint: Use your answers from parts (a) and (b) and (c).

e) Find the D-field (magnitude and direction) inside the disc.

Problem 3.6: (Magnetic field of a circular current loop)

Consider a line-current in the form of a circular loop of radius a and carrying a current I, as shown below. The loop is in the x-y plane. You need to find the magnetic field at the point P given by (0,0,z).



a) What is the x-component of the magnetic field at the location P, as shown in the figure above?

b) What is the y-component of the magnetic field at the location P, as shown in the figure above?

c) What is the z-component of the magnetic field at the location P, as shown in the figure above?