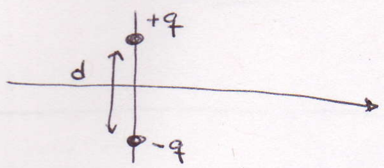
Homework 2 Solutions. Total (100 points).

Problem 2.1 (16 pts).

(+4) a). Method of Images. (+1).



$$\boxed{\phi(\vec{r}) = 0 \quad \text{for} \quad z < 0.} \quad (+1).$$

$$\phi(\vec{r}) = \phi_+(\vec{r}) + \phi_-(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right]. \quad (+1).$$

$$\boxed{\phi(\vec{r}) = \frac{q\,d\cos\theta}{4\pi\varepsilon_0 r^2} \quad \text{for} \quad z > 0.} \quad (+1).$$

(+4) b) $E(\vec{r}) = 0$ for $z < 0$. (+2).

$$E(\vec{r}) = -\nabla\phi(\vec{r})$$

$$= \frac{qd}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right] \quad (+2).$$

(+4) c). Upper. $\nabla\cdot E = \frac{\sigma}{\varepsilon_0}$.

$$E_{z1} - E_{z2} = \frac{-qd}{4\pi\cancel{\varepsilon_0}\left(r^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} = \frac{\sigma}{\cancel{\varepsilon_0}}$$

$$\boxed{\sigma = \frac{-qd}{4\pi\left(r^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}} \quad (+2).$$

Lower. $\boxed{\sigma = 0.}$ (+2).

d)

(+4). $Q = \int_0^{2\pi}\int_0^{\infty} \sigma(\bar{r})\, r \cdot dr\, d\theta$. (+2).

$= 2\pi \int_0^{\infty} \dfrac{-q\, d\, r}{4\pi\left(r^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\, dr$.

$u = r^2 + \left(\frac{d}{2}\right)^2$. $du = 2r\, dr$.

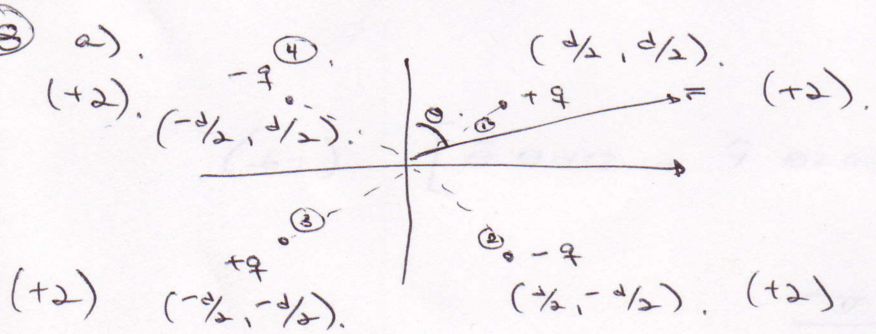
$= \dfrac{-q \cdot d}{2} \displaystyle\int_{\left(\frac{d}{2}\right)^2}^{\infty} \dfrac{du/2}{u^{3/2}}$.

$= \dfrac{-qd}{4}\left(-2/\sqrt{u}\right)_{(d/2)^2}^{\infty}$.

$\boxed{Q_{tot} = -q.}$ (+2).

Problem 2.2. (16 Points).

(+8) a).



(+8) b). $\phi(\bar{r}) = \phi_1(\bar{r}) + \phi_2(\bar{r}) + \phi_3(\bar{r}) + \phi_4(\bar{r})$. (+2)

$\phi_1(\bar{r}) = r - \frac{1}{\sqrt{2}} \cdot \cos(\theta - \pi/4)$. (+1)

$\phi_2(\bar{r}) = r - \frac{1}{\sqrt{2}} \sin(\theta - \pi/4)$. (+1)

$\phi_3(\bar{r}) = r + \frac{1}{\sqrt{2}} \cos(\theta - \pi/4)$. (+1)

$\phi_4(\bar{r}) = r + \frac{d}{\sqrt{2}} \sin(\theta - \pi/4)$. (+1).

$$\phi \simeq \frac{q d^2}{4\pi \epsilon_0 r^3} \sin(2\theta) \,. \qquad (+2).$$

☆ $\underline{\cos(\theta - \pi/2) = \sin(\theta)}$.

$\underline{\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)}$.

Problem 2.3. (18 points).

(+3).

a). (i) $E(\vec{r}) = \dfrac{q}{4\pi\epsilon_0 r^2}$, $0 < r < b$. $(+1)$

(ii) $E(\vec{r}) = 0$, $b < r < c$. $(+1)$

(iii) $E(\vec{r}) = \dfrac{q}{4\pi\epsilon_0 r^2}$, $c < r$. $(+1)$.

(+4)

b). Inner Surface. $\nabla \cdot E = \dfrac{\sigma}{\epsilon_0}$.

$\sigma = \dfrac{-q}{4\pi b^2}$, $(+1)$

$Q = -q$. $(+1)$.

Outer Surface.

$\sigma = \dfrac{q}{4\pi c^2}$. $(+1)$.

$Q = +q$ $(+1)$.

(+3).

c). (i) $0 < r < b$. $\phi(\vec{r}) = \dfrac{q}{4\pi\epsilon_0}\left[\dfrac{1}{r} - \dfrac{1}{b}\right]$ $(+1)$.

(ii) $b < r < c$. $\phi(\vec{r}) = 0$. $(+1)$

(iii) $c < r$ $\phi(\vec{r}) = 0$. $(+1)$.

☆ Use $\nabla^2 \phi = \dfrac{-\rho}{\epsilon_0} \longrightarrow \nabla^2\phi = 0$.

(+3) d). $E = -\nabla\phi$.

(i) $E(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0 r^2}$. $\quad 0 < r < b$. (+1)

(ii) $E(\vec{r}) = 0$. $\quad b < r < c$. (+1)

(iii) $E(\vec{r}) = 0$. $\quad c < r$. (+1)

(+4) e). Outer Surface: $\sigma = 0$, $Q = 0$. (+2)

Inner Surface: $\sigma = \dfrac{-q}{4\pi b^2}$ (+2)

$Q = -q$

(+1) f) Allowed Conductor to have non-zero net charge.

Problem 2.4. (15 points).

(+3) a) $\nabla^2\phi = 0$. for $a < r < b$ (+1)

$\phi(\vec{r}) = \dfrac{A}{r} + B$

$\phi(a) = C_0$. (+1)

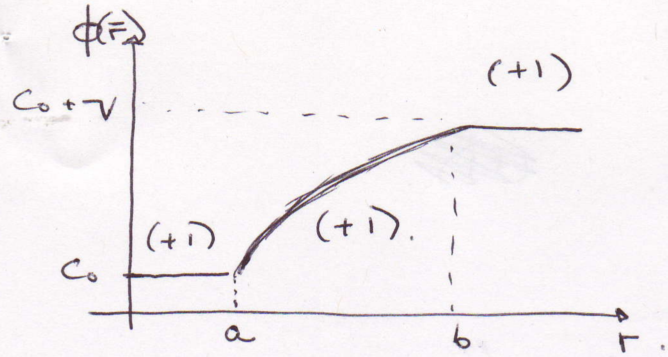
$\phi(b) = C_0 + V$.

$A = \dfrac{-V \cdot ab}{b-a}$.

$B = V \cdot \dfrac{b}{b-a} + C_0$.

$$\boxed{\phi(\vec{r}) = \frac{Vb}{b-a}\left[1 - \frac{a}{r}\right] + C_0 = \frac{Vab}{b-a}\left[\frac{1}{a} - \frac{1}{r}\right] \quad (+1)}$$

b) (+3)



(+3)

c). $E(\vec{r}) = -\frac{\partial}{\partial r} \cdot \phi(\vec{r})$

(i) $E(\vec{r}) = 0$ $\quad 0 < r < a.$ (+1)

(ii) $E(\vec{r}) = \frac{V ab}{b-a} \cdot \frac{-1}{r^2} \hat{r}$ $\quad a < r < b.$ (+1).

(iii) $E(\vec{r}) = 0.$ $\quad b < r < c$ (+1).

(+3) d). $\varepsilon_0 (E_{1\perp} - E_{2\perp}) = \sigma.$ (+1).

$\sigma_{\text{inner sphere}} = \frac{V b \cdot \varepsilon_0}{a(a-b)}.$ (+1).

$\sigma_{\text{outer sphere}} = \frac{V a \cdot \varepsilon_0}{b(b-a)}$ (+1).

(+3) e). $C = \frac{dQ}{dV}.$ (+1).

$dQ = d(A \cdot \sigma) = \cancel{\sigma\, dA} + A \cdot d\sigma.$ (+1).

Inner: $dQ = 4\pi a^2 \cdot d\sigma \longrightarrow \frac{dQ}{dV} = 4\pi a^2 \cdot \frac{d\sigma}{dV}.$

$= 4\pi a^{\cancel{2}} \cdot \frac{b\, \varepsilon_0}{\cancel{a}(a-b)} = \boxed{\frac{4\pi a b \varepsilon_0}{(a-b)}}.$

Outer: $dQ = 4\pi b^2 d\sigma \longrightarrow \boxed{\frac{dQ}{dV} = \frac{4\pi a b \varepsilon_0}{b-a}.}$ (+1).

Problem 2.5 (15 points).

(+5)

a). $\nabla^2 \phi = \frac{-\rho}{\varepsilon_0}$ for $\frac{-d}{2} < x < \frac{d}{2}$. $\cancel{(+1)}$.

$\phi(\frac{-d}{2}) = V$.

$\phi(\frac{d}{2}) = 0$. (+1).

$\phi(x) = Ax^2 + Bx + C$.

$A = \frac{-\rho}{2\varepsilon_0}$. (+1).

$B = -V/d$ (+1).

$C = \frac{V}{2} + \frac{\rho d^2}{8\varepsilon_0}$. (+1).

$$\phi(x) = \frac{-\rho}{2\varepsilon_0} \cdot x^2 - V/d\, x + \left(\frac{V}{2} + \frac{\rho d^2}{8\varepsilon_0} \right) \quad (+1).$$

(+2).

b). $E(x) = -\frac{\partial}{\partial x} \phi(x)$. (+1)

$E(x) = \frac{\rho x}{\varepsilon_0} + V/d$. (+1).

(+2)

c). Left: $\nabla \cdot E = \frac{\sigma}{\varepsilon_0}$.

$\sigma = -\rho \cdot \frac{d}{2} + \frac{V\varepsilon_0}{d}$ (+1).

Right: $\sigma = \frac{\rho d}{2} - \frac{V\varepsilon_0}{d}$ (+1).

(+3) d) $C = \frac{dQ}{dV} = \frac{A \cdot d\sigma}{dV} = \frac{\varepsilon_0 A}{d}$ (+3)

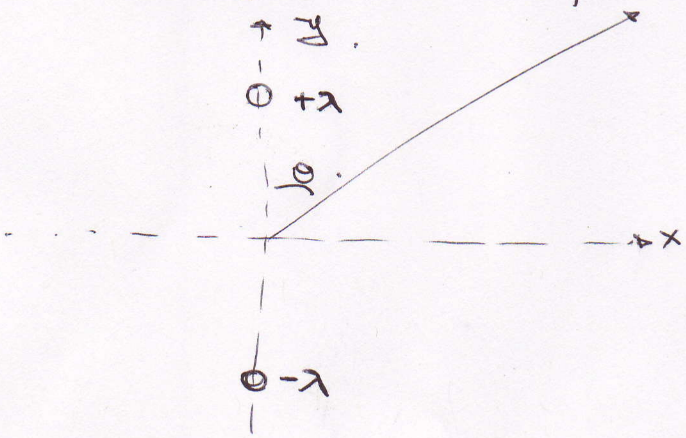
(+3) e) Use Gauss' Law (+2)

$Q = -\rho \cdot A \cdot d$ (+1)

Same & Opposite.

Problem 2.6 (20 points)

(+5) a)



(+5) b) $\phi(\bar{r}) = \phi_+(\bar{r}) + \phi_-(\bar{r})$ (+3)

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \left| \frac{r + \frac{d}{2}\cos\theta}{r - \frac{d}{2}\cos\theta} \right| \quad (+2)$$

(+5) c) $\underset{\text{on plate}}{\phi(\bar{r})} = 0$ for $\theta \longrightarrow \pi/2$ (+2)

$$\phi(d/2 \pm a) = V = \frac{\lambda}{2\pi\varepsilon_0} \ln \left| \frac{d \pm a}{a} \right|$$

$$\approx \frac{\lambda}{2\pi\varepsilon_0} \ln \left| \frac{d}{a} \right| \quad (+2)$$

$$\boxed{\lambda = \frac{2\pi\varepsilon_0 V}{\ln\left|\frac{d}{a}\right|}}$$ (+1).

+5

d) $\boxed{C = \frac{d\lambda}{dV} = \frac{2\pi\varepsilon_0}{\ln\left|\frac{d}{a}\right|} \quad (+5)}$