# ECE 303 - Homework #1 Solutions

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## 1.1

a) $\vec{\nabla}\phi = 2a(x\hat{x}+y\hat{y}+z\hat{z})$

b) Cylindrical Coords, $\phi = a(r^2+z^2)$, $\vec{\nabla}\phi = 2a(r\hat{r}+z\hat{z})$

c) Spherical Coords. $\phi = ar^2$ $\vec{\nabla}\phi = 2ar\hat{r}$

d) $\vec{\nabla}(\vec{\nabla}\cdot\phi) = 6a$

## 1.2

$\vec{F}(x,y,z) = x\hat{x}+y\hat{y}+z^2\hat{z}$

a) surface defined by $z=\frac{L}{2}$: $d\vec{a} = r\,d\phi\,dr\,\hat{z}$

$$\iint \vec{F}\cdot d\vec{a} = \int_0^a\int_0^{2\pi} \frac{L^2}{4}\,r\,d\phi\,dr = \frac{L^2}{4}\pi a^2$$

surface defined by $z=-\frac{L}{2}$: $d\vec{a} = r\,d\phi\,dr\,(-\hat{z})$

$$\iint \vec{F}\cdot d\vec{a} = -\frac{L^2}{4}\pi a^2$$

surface defined by $r=a$: $d\vec{a} = a\,d\phi\,dz\,\hat{r}$

For convenience express $\vec{F}$ in cylindrical coordinates: $\vec{F} = r\hat{r}+z^2\hat{z}$

$$\iint \vec{F}\cdot d\vec{a} = \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^{2\pi} a^2\,d\phi\,dz = 2\pi a^2 L.$$

Adding up all contributions:

$$\oiint \vec{F}\cdot d\vec{a} = 2\pi a^2 L$$

1.2 cont'd

b) $\vec{\nabla}\cdot\vec{F} = 2 + 2z$

c) $\oiiint \vec{\nabla}\cdot\vec{F}\,dV = \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^a \int_0^{2\pi} (2+2z)\, r\, d\phi\cdot dr\, dz = 2\pi a^2 L$

d) from a) and c) $\oiint \vec{F}\cdot d\vec{a} = \oiiint \vec{\nabla}\cdot\vec{F}\,dV = 2\pi a^2 L$

1.3 $\quad \vec{F}(x,y,z) = z\hat{r} + r\hat{\phi} + \hat{z}$

a) contour defined by $r=a$, $z=0$ $\quad d\vec{s} = a\,d\phi\,\hat{\phi}$

$$\oint \vec{F}\cdot d\vec{s} = \int_0^{2\pi} a^2\, d\phi = 2\pi a^2$$

b) $\vec{\nabla}\times\vec{F} = \hat{\phi} + 2\hat{z}$

c) $d\vec{a} = r\,d\phi\,dr\,\hat{z}$ so: $\iint (\vec{\nabla}\times\vec{F})\cdot d\vec{a} = \int_0^a \int_0^{2\pi} 2r\,dr\,d\phi = 2\pi a^2$

d) From a) and c) $\oint \vec{F}\cdot d\vec{s} = \iint (\vec{\nabla}\times\vec{F})\cdot d\vec{a} = \underline{2\pi a^2}$

e) For convenience express $\vec{\nabla}\times\vec{F}$ in spherical coordinates:

$$\vec{\nabla}\times\vec{F} = \hat{\phi} + 2(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

on the surface: $d\vec{a} = a^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

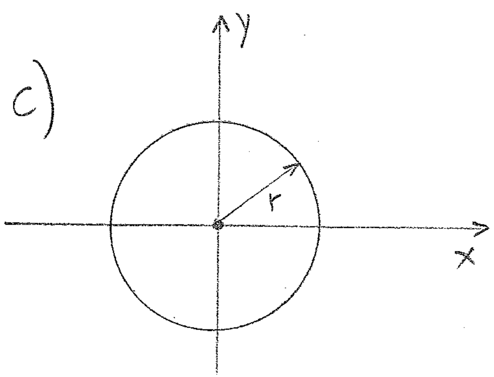
so $\iint (\vec{\nabla}\times\vec{F})\cdot d\vec{a} = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2a^2 \sin\theta\cos\theta\, d\theta\, d\phi = \underline{2\pi a^2}$

1.4

a) The charge distribution has complete symmetry w.r.t. $z=0$ so, if there was an electric field in the $+\hat{z}$ direction there should also be an equal field in the $-\hat{z}$ direction. This is impossible since a field cannot point in two directions at the same time. Therefore the z-component of the E field must be zero

b) Suppose that, looking at the line charge from the $+z$ axis, we can see a field with a component in the $+\hat{\phi}$ direction (counterclockwise). Then, looking at the line charge from the $-z$ axis, we should also see a field in the $-\hat{\phi}$ direction (clockwise). But, since the line charge is the same whether we look at it from the $+z$ axis or the $-z$ axis, this would imply that we have a field pointing in the $+\hat{\phi}$ and in the $-\hat{\phi}$ directions at the same time. This is impossible, thus the $\hat{\phi}$-component of the electric field must be zero.

c)




Take a cylindrical surface of radius $r$ and length $\ell$ into the page, with the line charge along the cylinder axis. From a) and b) the electric field only has a radial component $E_r$

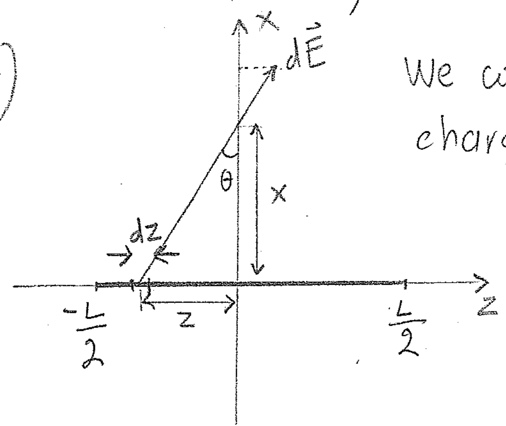
From Gauss' Law: Total electric flux coming out of the cylinder must be equal to the total charge enclosed by the cylinder.

$$\left.\begin{array}{l} \text{Flux} = \oint\!\!\oint \varepsilon_0 \vec{E}\cdot d\vec{a} = \varepsilon_0 E_r 2\pi r \ell \\ \text{Total charge} = \lambda \ell \end{array}\right\} \Rightarrow E_r = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{and} \quad \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

## 1.5

a) On the x-axis the z-component of the electric field is zero due to symmetry

b)




We consider the line charge as consisting of elemental charges of $\lambda dz$ Coulombs. Each such charge contributes a field $d\vec{E}$ with magnitude:

$$dE\Big|_{z=0} = \frac{\lambda dz}{4\pi\varepsilon_0 (x^2+z^2)}$$

The projection of $d\vec{E}\Big|_{z=0}$ on the x-axis is $dE_x\Big|_{z=0} = dE\Big|_{z=0} \cdot \cos\theta$

$$\Rightarrow \quad dE_x\Big|_{z=0} = \frac{\lambda dz}{4\pi\varepsilon_0 (x^2+z^2)} \frac{x}{\sqrt{x^2+z^2}} = \frac{\lambda x \, dz}{4\pi\varepsilon_0 (x^2+z^2)^{3/2}}$$

The total field $E_x\Big|_{z=0}$ is the sum of all contributions along the line charge:

$$E_x\Big|_{z=0} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\lambda x}{4\pi\varepsilon_0 (x^2+z^2)^{3/2}} dz = \frac{\lambda}{4\pi\varepsilon_0 x} \frac{L}{\sqrt{x^2+\frac{L^2}{4}}} \qquad x>0$$

c) For $L \gg x$ $\quad E_x\Big|_{z=0} = \frac{\lambda}{2\pi\varepsilon_0 x}$

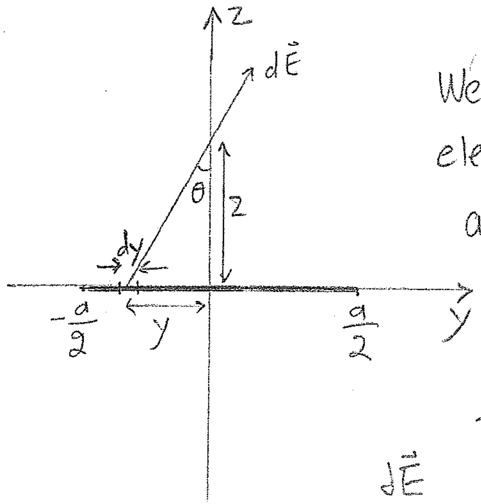
d) this result agrees with the answer in 1.4c)

## 1.6

a) By symmetry the x-component of the electric field on the z-axis is zero

b) By symmetry the y-component of the field on the z-axis is zero

c)




We consider the square sheet of charge as consisting of elemental line charges of length a (into the page) and $\sigma . dy$ Coulombs per unit length

Each of these line charges contributes a field $d\vec{E}$ on the z-axis. The magnitude of $d\vec{E}$ is given by the answer in 1.5b) If we substitute:

$$\lambda \to \sigma dy, \quad x^2 = y^2 + z^2, \quad L \to a$$

i.e.

$$dE\Big|_{y=0} = \frac{\sigma dy}{4\pi\varepsilon_0 \sqrt{y^2+z^2}} \frac{a}{\sqrt{y^2+z^2+\frac{a^2}{4}}}$$

The projection of $d\vec{E}\Big|_{y=0}$ on the z-axis is $dE_z\Big|_{y=0} = dE\Big|_{y=0} \cos\theta$

$$\Rightarrow dE_z\Big|_{y=0} = \frac{\sigma dy\, az}{4\pi\varepsilon_0 (y^2+z^2)\sqrt{y^2+z^2+\frac{a^2}{4}}}$$

Summing up all contributions: $E_z\Big|_{y=0} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{\sigma a z}{4\pi\varepsilon_0 (y^2+z^2)\sqrt{y^2+z^2+\frac{a^2}{4}}} dy$

Using the hint: $E_z\Big|_{y=0} = \frac{\sigma}{2\pi\varepsilon_0}\left[\pi - \tan^{-1}\left(\frac{4a^2 z\sqrt{2a^2+4z^2}}{a^4-16z^4-8a^2z^2}\right)\right]$

## 1.6 cont'd

d) For $a >> z$ $\tan^{-1}\{ \quad \} \to 0$

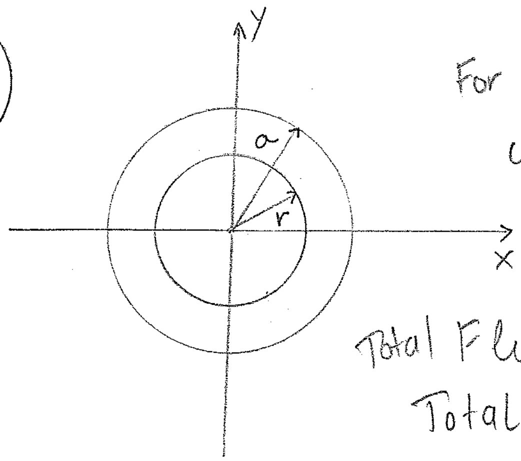
and:

$$E_z\Big|_{y=0} = \frac{\sigma}{2\varepsilon_0}$$ i.e. the field of an infinite sheet of charge

## 1.7

Due to symmetry the electric field only has a radial component $E_r$

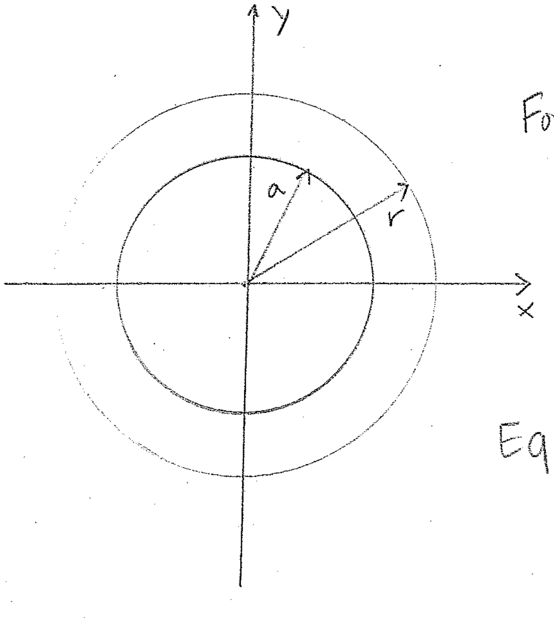
a)




For $0 \le r \le a$ we apply Gauss' Law using a cylinder a radius $r \le a$ and length $l$ into the page

Total Flux coming out of the cylinder $= \varepsilon_0 E_r 2\pi r l$
Total charge enclosed by the cylinder $= \rho \pi r^2 l$

Equating the two: $E_r = \dfrac{\rho r}{2\varepsilon_0}$ , $0 \le r \le a$




For $a \le r \le \infty$ we take a cylinder of radius $r \ge a$

Total Flux out of the cylinder $= \varepsilon_0 E_r 2\pi r l$
Total charge in the cylinder $= \rho \pi a^2 l$

Equating the two: $E_r = \dfrac{\rho a^2}{2\varepsilon_0 r}$ $a \le r \le \infty$