School of Electrical and Computer Engineering, Cornell University

ECE 303: Electromagnetic Fields and Waves

Fall 2007

Homework 1 **Due on Aug. 31, 2007 by 5:00 PM**

Reading Assignments:

i) Review the material on cartesian, cylindrical, and spherical co-ordinate systems from your favorite freshman calculus book. Make sure you are comfortable in using these co-ordinate systems.

ii) Relevant sections of the online *Haus and Melcher* book for this week are 2.0-2.6, 3.2, 3.3. Note that the book contains more material than you are responsible for in this course. Determine relevance by what is covered in the lectures and the recitations.

Problem 1.1: (warm up)

Consider a scalar quantity ϕ given by the expression,

$$\phi(x,y,z) = a\left(x^2 + y^2 + z^2\right)$$

Where a is some constant.

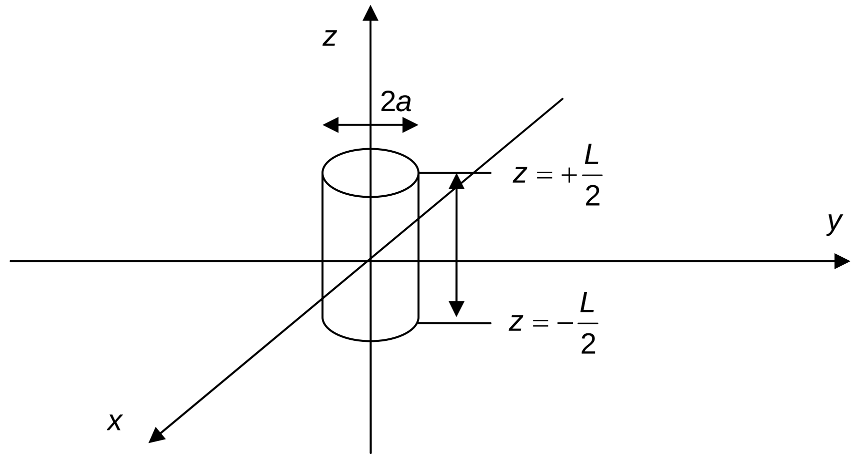
a) Find the gradient $\vec{\nabla}\phi$ of the scalar ϕ in the Cartesian co-ordinate system.
b) Express the scalar ϕ given above in variables of the cylindrical co-ordinate system and then find the gradient $\vec{\nabla}\phi$ in the cylinderical co-ordinate system.
c) Express the scalar ϕ given above in variables of the spherical co-ordinate system and then find the gradient $\vec{\nabla}\phi$ in the spherical co-ordinate system.
d) Find the divergence of the gradient for the scalar ϕ (i.e. first find the gradient vector and then find the divergence of the gradient vector). Note: The divergence of the gradient, written as $\vec{\nabla}\bullet\left(\vec{\nabla}\phi\right)$, is also more commonly denoted by the Laplacian operator ∇^2, i.e. $\vec{\nabla}\bullet\left(\vec{\nabla}\phi\right) = \nabla^2\phi$.

Problem 1.2: (basic vector calculus review)

Consider a vector field $\vec{F}$ given by the expression:

$$\vec{F}(x,y,z) = x\ \hat{x} + y\ \hat{y} + z^2\ \hat{z}$$

a) Find the total flux associated with the vector $\vec{F}$ coming out of a closed surface which is in the form of a cylinder and shown below in the figure. The length of the cylinder is L, the diameter is $2a$, and the axis of the cylinder is the z-axis. In other words, you are supposed to directly evaluate the surface integral: $\oint\!\!\oint \vec{F} \bullet d\vec{a}$



b) Calculate the divergence $\vec{\nabla} \bullet \vec{F}$ of the vector $\vec{F}$.

c) Calculate directly the volume integral: $\iiint \vec{\nabla} \bullet \vec{F}\, dV$ where the integral is over the volume enclosed by the cylinder shown in the figure above.

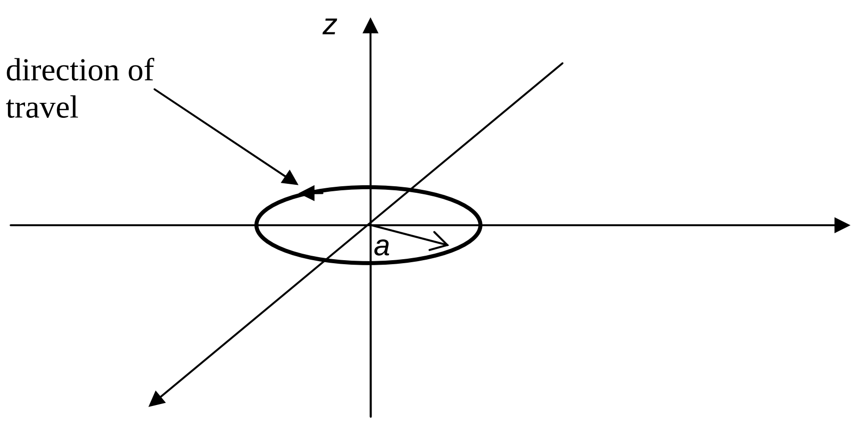
d) Using your results from part (a) and part (c) verify Gauss' theorem (or the divergence theorem): $\iiint \vec{\nabla} \bullet \vec{F}\, dV = \oint\!\!\oint \vec{F} \bullet d\vec{a}$

Problem 1.3: (basic vector calculus review)

Consider a vector field given in the cylindrical co-ordinate system by the expression:

$$\vec{F}(x,y,z) = z\,\hat{r} + r\,\hat{\phi} + \hat{z}$$

a) Evaluate directly the line integral $\oint \vec{F} \bullet d\vec{s}$ of the vector $\vec{F}$ for a closed **circular** contour located entirely in the plane $z = 0$ and of radius a, going in the direction indicated by the arrow in the figure below.

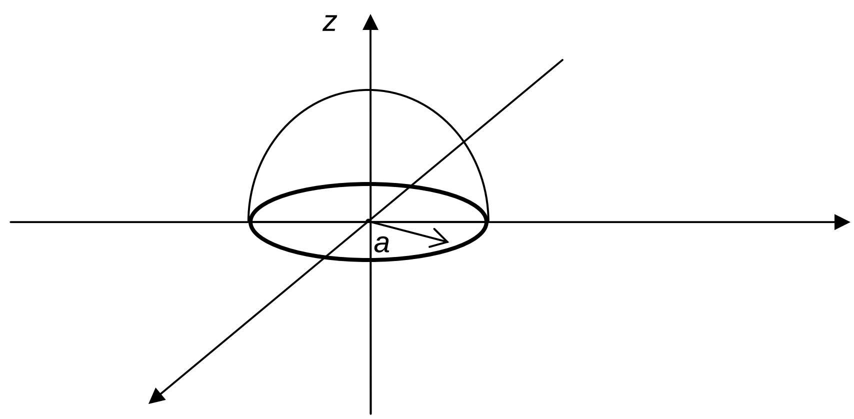


b) Calculate the curl of the vector $\vec{F}$, i.e. $\vec{\nabla} \times \vec{F}$.

c) Calculate directly the flux of the curl of the vector $\vec{F}$ through the circular surface that is located entirely in the plane $z = 0$ plane and is bounded by the circular contour shown in the figure above. You are supposed to directly evaluate the surface integral: $\iint \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{a}$. Take the vector $d\vec{a}$ to point in the $+\hat{z}$ direction.

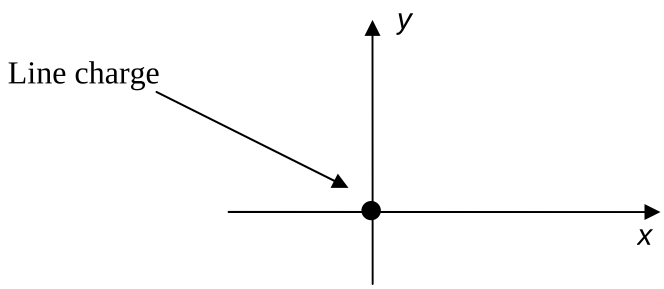
d) Using your results from part (a) and part(c) verify Stokes theorem: $\iint \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{a} = \oint \vec{F} \bullet d\vec{s}$

e) Now a more challenging problem. Calculate by any method the flux of the curl of the vector $\vec{F}$ through the surface which is the upper half of a spherical shell (centered at the origin) as shown in the figure below.



Problem 1.4: (electrostatics of a line charge)

Consider an infinite line of charge that carries λ Coulombs of positive charge per unit length. The line charge is oriented along the z-axis which is perpendicular to the plane of this paper.



a) Use symmetry to explain why there cannot be a component of the electric field in the $\pm$z-directions. No points will be awarded for wrong or bad explanations.

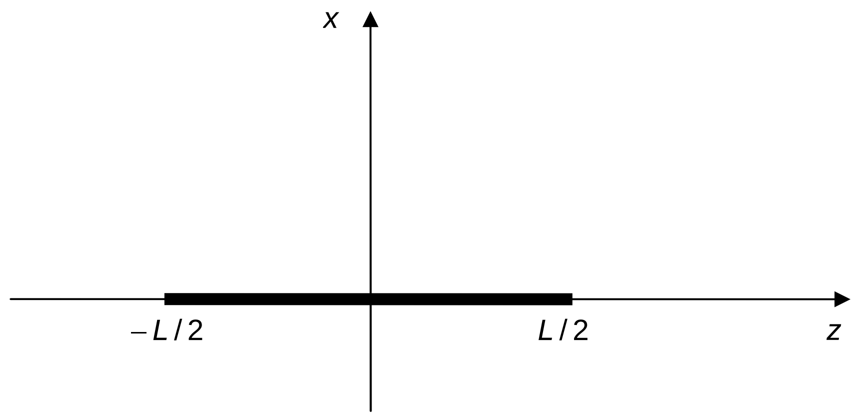
b) Use symmetry to explain why there cannot be a component of the electric field in the $\hat{\phi}$ direction.

c) Use Gauss' Law and find the direction and magnitude of the electric field as a function of the radial distance r from the line charge. Indicate the direction by an appropriate unit vector.

Problem 1.5: (electrostatics of a line charge)

Consider a line of charge of length L that carries λ Coulombs of positive charge **per unit length**, as shown in the figure below. The line charge is oriented along the z-axis.

a) Find the z-component of the electric field as a function of position x along the x-axis (where $z = 0$).



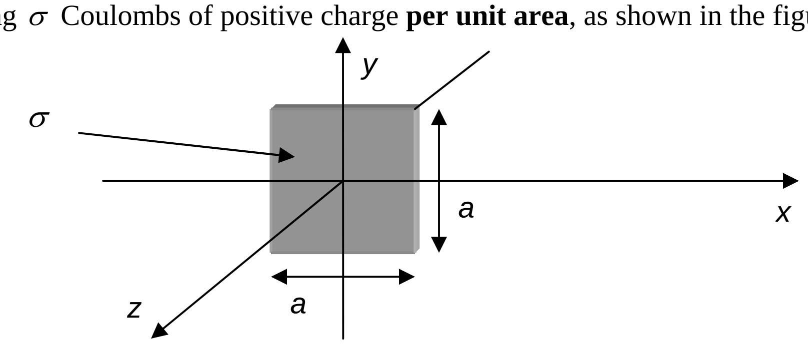
b) Find the x-component of the electric field as a function of position x along the x-axis (where $z = 0$). Hint: you will have to set up an integral and evaluate it.

c) In your answer to part (b), take the limit that the length of the line charge is much much larger than the distances x that you are interested in (in other words, you are trying to find a formula for the electric field for distances real close to the line charge). What is the answer?

d) Compare your answer in part (c) to what you obtained in problem 1.4(c).

Problem 1.6: (electrostatics of a charged plate)

Consider a square charged plate (of almost zero thickness) with sides of length a, lying entirely in the x-y plane, and containing σ Coulombs of positive charge **per unit area**, as shown in the figure below.



a) Find the x-component of the electric field as a function of position z along the z-axis (where $x, y = 0$).

b) Find the y-component of the electric field as a function of position z along the z-axis (where $x, y = 0$).

c) Find the z-component of the electric field as a function of position z along the z-axis (where $x, y = 0$). Hint: you would have to set up an integral and evaluate it. Use your results from problem 1.5.

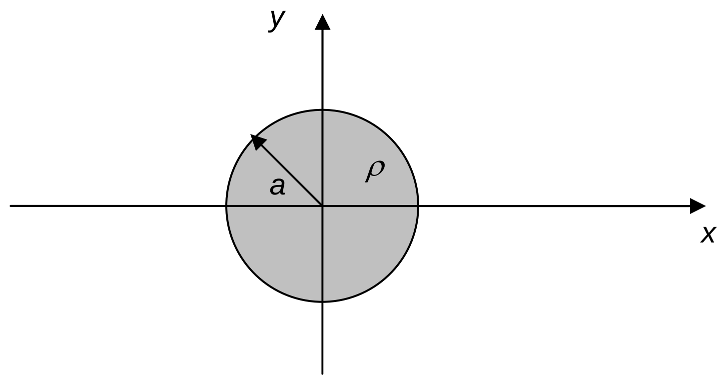
d) In your result for part (c) take the limit that the size of the plate is much much larger than the distances z that you are interested in (in other words, you are trying to find a formula for the z-component of the electric field for distances z real close to the charge plate). What is the answer?

Hint: The following integral may prove helpful:

$$\int_{-a/2}^{a/2} \frac{dy}{\left(y^2+z^2\right)\sqrt{y^2+z^2+a^2/4}} = \frac{2}{a\,z}\left[\pi - \tan^{-1}\left(\frac{4a^2 z\sqrt{2a^2+4z^2}}{a^4-16z^4-8a^2z^2}\right)\right]$$

Problem 1.7: (electrostatics of a charged cylinder)

Consider an infinitely long charged cylinder that carries ρ Coulombs of positive charge **per unit volume**. The cylinder is oriented along the z-axis which is perpendicular to the plane of this paper.



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a) Use Gauss' Law and find the direction and magnitude of the electric field as a function of the radial distance r for $0 \le r \le a$.

b) Use Gauss' Law and find the direction and magnitude of the electric field as a function of the radial distance r for $a \le r \le \infty$.