Lecture 9

Magnetoquasistatics

In this lecture you will learn:

- **Basic Equations of Magnetoquasistatics**
- **The Vector Potential**
- **The Vector Poisson's Equation**
- **The Biot-Savart Law**
- **Magnetic Field of Some Simple Current Carrying Elements**
- **The Magnetic Current Dipole**

Equations of Magnetoquasistatics

Equations of Electroquasistatics

$$\nabla . \varepsilon_o \vec{E} = \rho(\vec{r},t)$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \varepsilon_o \vec{E}}{\partial t}$$

- **Electric fields are produced by only electric charges**
- **Once the electric field is determined, the magnetic field can be found by the last equation above**

Equations of Magnetoquasistatics

$$\nabla . \mu_o \vec{H} = 0$$

$$\nabla \times \vec{H} = \vec{J}(\vec{r},t)$$

$$\nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$

- **Magnetic fields are produced by only electric currents**
- **Once the magnetic field is determined, the electric field can be found by the last equation above**
- **Currents in magnetoquasistatics are solenoidal (i.e. with zero divergence)**

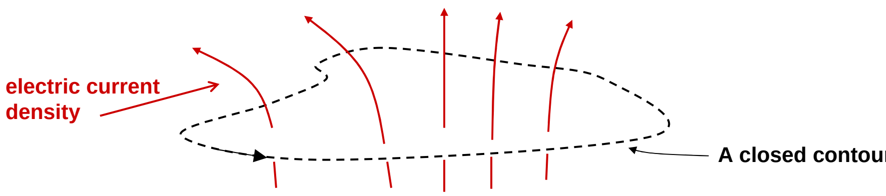
$$\nabla . \vec{J}(\vec{r},t) = \nabla . \left(\nabla \times \vec{H}\right) = 0$$

In magnetoquasistatics the source of the magnetic field is electrical current

Ampere's Law for Magnetoquasistatics

$$\oint \vec{H}\,.\,d\vec{s} = \iint \vec{J}\,.\,d\vec{a}$$

$$\nabla \times \vec{H} = \vec{J}$$



Ampere's Law: The line integral of magnetic field over a closed contour is equal to the total current flowing through that contour

Right Hand Rule: The positive directions for the surface normal vector and of the contour are related by the right hand rule

Magnetic Field of an Infinite Line-Current

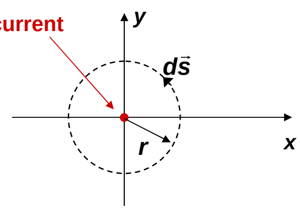
Consider an infinitely long line-current carrying a total current *I* in the +*z*-direction, as shown below

Use ampere's law on the closed contour shown by the dashed line:

$$\oint \vec{H}\,.\,d\vec{s} = \iint \vec{J}\,.\,d\vec{a}$$

$$\Rightarrow \quad (2\pi r) H_\phi(r) = I$$

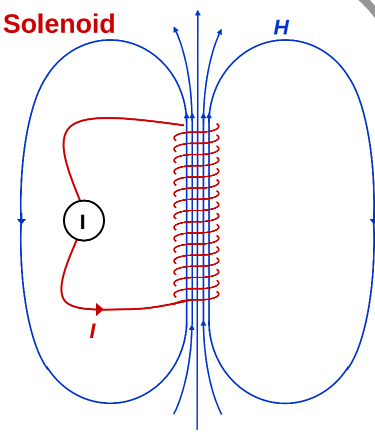
$$\Rightarrow \quad H_\phi(r) = \frac{I}{2\pi r}$$



Working in the cylindrical coordinates

Magnetic field is entirely in the $\hat{\phi}$ direction and falls off as ~1/*r* from the line-current

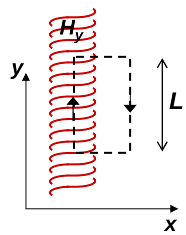
Magnetic Field of a Solenoid



Consider a solenoid with *N* turns per unit length and carrying a current *I*

Assumptions:

- The magnetic field inside the solenoid is uniform and strong
- There is a fringing field outside the solenoid which is very weak and may be neglected



$$\oint \vec{H}\,.\,d\vec{s} = \iint \vec{J}\,.\,d\vec{a}$$
$$\Rightarrow \quad L\,H_y = (LN)\,I$$
$$\Rightarrow \quad H_y = NI$$

The Vector Potential - I

A Vector Identity:

For any vector $\vec{F}$ the divergence of the curl is always zero:

$$\nabla\,.\left(\nabla \times \vec{F}\right) = 0$$

The Vector Potential:

In magnetoquasistatics the divergence of the B-field is always zero:

$$\nabla\,.\left(\vec{B}\right) = \nabla\,.\left(\mu_o\,\vec{H}\right) = 0$$

So one may represent the B-field as the curl of another vector:

$$\vec{B} = \mu_o\,\vec{H} = \nabla \times \vec{A}$$

$\vec{A}$ is called the vector potential

$$\Rightarrow \quad \nabla\,.\,\vec{B} = \nabla\,.\,\mu_o\,\vec{H} = \nabla\,.\left(\nabla \times \vec{A}\right) = 0$$

ECE 303 – Fall 2007 – Farhan Rana – Cornell University

The Vector Potential - II

In electroquasistatics we had: $\nabla \times \vec{E} = 0$

Therefore we could represent the E-field by the scalar potential: $\vec{E} = -\nabla \phi$

In magnetoquasistatics we have: $\nabla . (\vec{B}) = \nabla . (\mu_o \vec{H}) = 0$

Therefore we can represent the B-field by the vector potential:

$$\vec{B} = \mu_o \vec{H} = \nabla \times \vec{A}$$

A vector field can be specified (up to a constant) by specifying its curl and its divergence

Our definition of the vector potential $\vec{A}$ is not yet unique – we have only specified its curl

For simplicity we fix the divergence of the vector potential $\vec{A}$ to be zero:

$$\nabla . \vec{A} = 0$$

Differential Equation for The Vector Potential

Start from Ampere's law in differential form: $\nabla \times \vec{H} = \vec{J}$

Use: $\mu_o \vec{H} = \nabla \times \vec{A}$

To get: $\nabla \times (\nabla \times \vec{A}) = \mu_o \vec{J}$

A Vector Identity: $\nabla \times (\nabla \times \vec{F}) = \nabla (\nabla . \vec{F}) - \nabla^2 \vec{F}$

$\nabla (\nabla . \vec{A}) - \nabla^2 \vec{A} = \mu_o \vec{J}$ } ⟶ Use: $\nabla . \vec{A} = 0$

$\Rightarrow \boxed{\nabla^2 \vec{A} = -\mu_o \vec{J}}$ ⟶ The differential equation for the vector potential (also called the Vector Poisson's Equation)

This is in fact 3 different equations (one for each component of $\vec{A}$)

$$\nabla^2 A_x = -\mu_o J_x \qquad \nabla^2 A_y = -\mu_o J_y \qquad \nabla^2 A_z = -\mu_o J_z$$

The Superposition Principle for The Vector Potential

- Suppose for the current distribution $\vec{J}_1(\vec{r})$ we have found the vector potential $\vec{A}_1(\vec{r})$

- Suppose for some other current distribution $\vec{J}_2(\vec{r})$ we have also found the vector potential $\vec{A}_2(\vec{r})$

- Then the vector potential $\left(\vec{A}_1(\vec{r})+\vec{A}_2(\vec{r})\right)$ is the solution for the current distribution $\left(\vec{J}_1(\vec{r})+\vec{J}_2(\vec{r})\right)$

Proof:

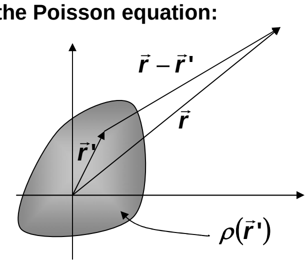
$$\nabla^2\vec{A}_1(\vec{r}) = -\mu_o\,\vec{J}_1(\vec{r}) \quad + \quad \nabla^2\vec{A}_2(\vec{r}) = -\mu_o\,\vec{J}_2(\vec{r}) \quad =$$

$$\nabla^2\left(\vec{A}_1(\vec{r})+\vec{A}_2(\vec{r})\right) = -\mu_o\left(\vec{J}_1(\vec{r})+\vec{J}_2(\vec{r})\right)$$

Recall the Superposition Integral for the Potential

In the most general scenario, one has to solve the Poisson equation:

$$\nabla^2\phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o}$$



We know that the solution for a point charge sitting at the origin:

$$\phi(\vec{r}) = \frac{q}{4\pi\,\varepsilon_o\,r}$$

To find the potential at any point one can sum up the contributions from different portions of a charge distribution treating each as a point charge

$$\phi(\vec{r}) = \iiint \frac{\rho(\vec{r}\,')}{4\pi\,\varepsilon_o\,|\vec{r}-\vec{r}\,'|}\,dV' \qquad dV' = dx'\,dy'\,dz'$$

A formal solution of the vector differential equation is the vector superposition integral:

$$\nabla^2\vec{A} = -\mu_o\,\vec{J} \quad \Longrightarrow \quad \vec{A}(\vec{r}) = \iiint \frac{\mu_o}{4\pi}\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV'$$

The Biot-Savart Law - I

Start from the superposition integral for the vector potential:

$$\vec{A}(\vec{r}) = \iiint \frac{\mu_o}{4\pi} \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \, dV'$$

Now find the magnetic field:

$$\vec{H} = \frac{\nabla \times \vec{A}}{\mu_o}$$

$$\vec{H}(\vec{r}) = \nabla \times \left[\iiint \frac{\vec{J}(\vec{r}\,')}{4\pi \, |\vec{r}-\vec{r}\,'|} \, dV' \right]$$

A vector Identity:

$$\nabla \times (\varphi \vec{F}) = \varphi (\nabla \times \vec{F}) + (\nabla \varphi) \times \vec{F}$$

$$\vec{H}(\vec{r}) = \iiint \frac{1}{4\pi} \nabla \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) \times \vec{J}(\vec{r}\,') \, dV'$$

Recall that: $\nabla \left(\frac{1}{r} \right) = -\frac{\hat{r}}{|\vec{r}|^2}$

$$\nabla \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) = -\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = -\frac{n_{\vec{r}\,'\to\vec{r}}}{|\vec{r}-\vec{r}\,'|^2}$$

$$\vec{H}(\vec{r}) = \iiint \frac{\vec{J}(\vec{r}\,') \times \hat{n}_{\vec{r}\,'\to\vec{r}}}{4\pi |\vec{r}-\vec{r}\,'|^2} \, dV'$$

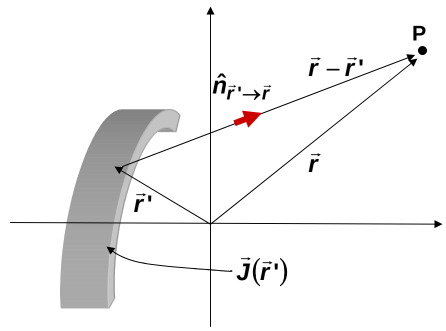
← **Biot-Savart Law**

$\hat{n}_{\vec{r}\,'\to\vec{r}}$ is a unit vector directed from point $\vec{r}\,'$ on the current source to the observation point $\vec{r}$

The Biot-Savart Law - II

Biot-Savart Law:

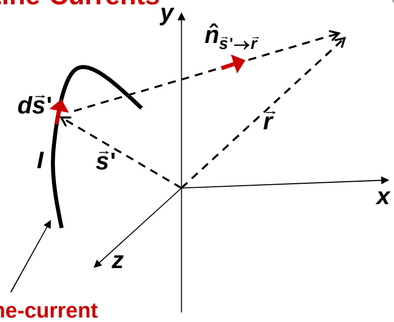
$$\vec{H}(\vec{r}) = \iiint \frac{\vec{J}(\vec{r}\,') \times \hat{n}_{\vec{r}\,'\to\vec{r}}}{4\pi |\vec{r}-\vec{r}\,'|^2} \, dV'$$



$\hat{n}_{\vec{r}\,'\to\vec{r}}$ is a unit vector directed from point $\vec{r}\,'$ on the current source to the observation point $\vec{r}$

Biot-Savart Law for Line-Currents

Need to find a formula that gives the total magnetic field at a point due to a current carrying wire as a superposition of magnetic field contributions from all the small pieces of the wire



line-current carrying a current I

Start from the Biot-Savart law:

$$\vec{H}(\vec{r}) = \iiint \frac{\vec{J}(\vec{r}\,')\times\hat{n}_{\vec{r}\,'\to\vec{r}}}{4\pi\left|\vec{r}-\vec{r}\,'\right|^2}\,dV'$$

$$= \iiint \frac{\vec{J}(\vec{r}\,')\times\hat{n}_{\vec{r}\,'\to\vec{r}}}{4\pi\left|\vec{r}-\vec{r}\,'\right|^2}\,da'\,ds' = \iiint \frac{\vec{I}(\vec{r}\,')\times\hat{n}_{\vec{r}\,'\to\vec{r}}}{4\pi\left|\vec{r}-\vec{r}\,'\right|^2}\,ds' \longrightarrow$$

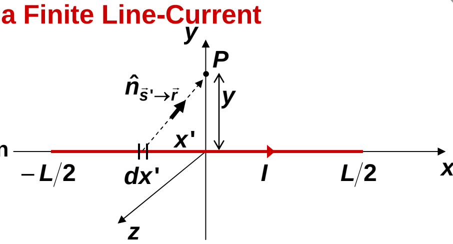
Integrate over the cross-section area of the wire to get the total current carried by the wire

$$= \frac{I}{4\pi}\int \frac{d\vec{s}\,'\times\hat{n}_{\vec{s}'\to\vec{r}}}{\left|\vec{r}-\vec{s}\,'\right|^2}$$

Magnetic Field of a Finite Line-Current

Consider a line-current of length L with current I in the +x-direction.

Find the magnetic field at the point P on the y-axis (as shown)



Use the Biot-Savart law:

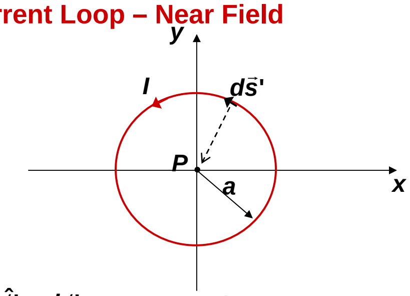
$$\vec{H}(0,y,0) = \frac{I}{4\pi}\int \frac{d\vec{s}\,'\times\hat{n}_{\vec{s}'\to\vec{r}}}{\left|\vec{r}-\vec{s}\,'\right|^2} \quad \left\{ \begin{array}{l} d\vec{s}\,' = \hat{x}\,dx' \\ \left|\vec{r}-\vec{s}\,'\right|^2 = x'^2 + y^2 \end{array} \right. \qquad d\vec{s}\,'\times\hat{n}_{\vec{s}'\to\vec{r}} = \hat{z}\,\frac{y\,dx'}{\sqrt{x'^2+y^2}}$$

$$= \hat{z}\,\frac{I}{4\pi}\int_{-L/2}^{L/2}\frac{y\quad dx'}{\left(x'^2+y^2\right)^{3/2}} = \hat{z}\,\frac{I}{2\pi\,y}\left[\frac{L/2}{\sqrt{y^2+(L/2)^2}}\right]$$

As $L\to\infty$, $\vec{H}(0,y,0)\to\hat{z}\,\dfrac{I}{2\pi\,y}$ .....recover the previous result

Magnetic Field of a Current Loop – Near Field

Consider a line-current in the form of a circular loop of radius *a* and carrying a current *I*, as shown in the figure



Find the magnetic field at the point *P* in the center of the loop

Use the Biot-Savart law:

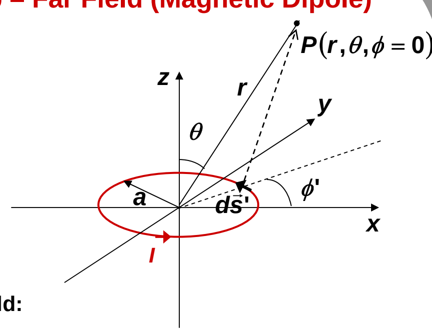
$$\vec{H}(0,0,0) = \frac{I}{4\pi}\int \frac{d\vec{s}' \times \hat{n}_{\vec{s}'\to\vec{r}}}{|\vec{r}-\vec{s}'|^2}$$

$$d\vec{s}' = \hat{\phi}'\, a\, d\phi' \qquad |\vec{r}-\vec{s}'|^2 = a^2 \qquad d\vec{s}'\times\hat{n}_{\vec{s}'\to\vec{r}} = \hat{z}\, a\, d\phi'$$

$$= \hat{z}\frac{I}{4\pi}\int_0^{2\pi}\frac{a\, d\phi'}{a^2} = \hat{z}\frac{I}{2a}$$

Magnetic Field of a Current Loop – Far Field (Magnetic Dipole)

Consider a line-current in the form of a circular loop of radius *a* and carrying a current *I*, as shown in the figure (*r* >> *a*)



Find the magnetic field at the point *P* far far away from the loop

A small current loop such as this is a magnetic dipole

Use the superposition integral for the A-field:

$$\vec{A}(r,\theta,\phi=0) = \iiint \frac{\mu_o\, \vec{J}(\vec{r}')}{4\pi\, |\vec{r}-\vec{s}'|}\, dV' = \frac{\mu_o I}{4\pi}\int \frac{d\vec{s}'}{|\vec{r}-\vec{s}'|}$$

→ Integrate over the cross-section of the wire

$$d\vec{s}' = \hat{\phi}'\, a\, d\phi' \longrightarrow \hat{\phi}' = \cos(\phi')\,\hat{y} - \sin(\phi')\,\hat{x}$$

→ Be careful – tricky integral – the unit vector is changing directions within the integral

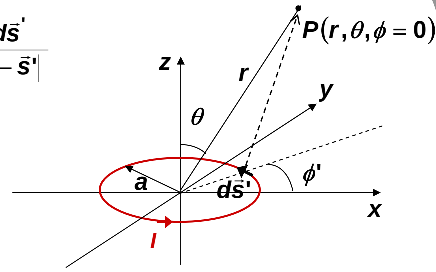
$$|\vec{r}-\vec{s}'| \approx r - a\sin(\theta)\cos(\phi')$$

Magnetic Field of a Current Loop – Far Field (Magnetic Dipole)

$$\vec{A}(r,\theta,\phi=0) = \iiint \frac{\mu_o \vec{J}(\vec{r}\,')}{4\pi\,|\vec{r}-\vec{s}\,'|}\,dV' = \frac{\mu_o I}{4\pi}\int \frac{d\vec{s}\,'}{|\vec{r}-\vec{s}\,'|}$$

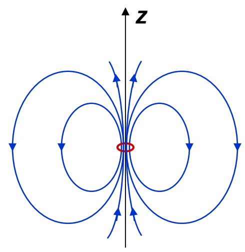
$$\approx \frac{\mu_o I}{4\pi}\int_0^{2\pi} \frac{[\cos(\phi')\hat{y} - \sin(\phi')\hat{x}]\,a\,d\phi'}{r - a\sin(\theta)\cos(\phi')}$$

$$\approx \frac{\mu_o I\left(\pi a^2\right)}{4\pi}\frac{\sin(\theta)}{r^2}\hat{y}$$



More generally:

$$\vec{A}(r,\theta,\phi) \approx \frac{\mu_o I\left(\pi a^2\right)}{4\pi}\frac{\sin(\theta)}{r^2}\hat{\phi}$$

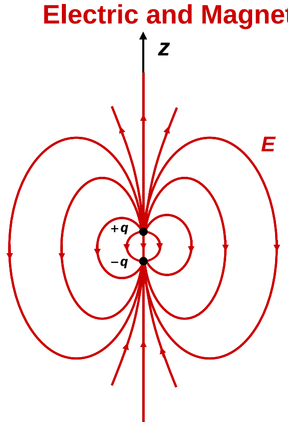


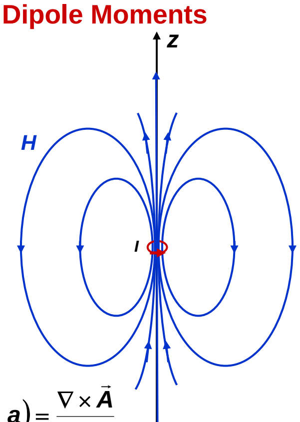
And:

$$\vec{H}(r,\theta,\phi) = \frac{\nabla\times\vec{A}}{\mu_o}$$

$$\approx \frac{\mu_o I\left(\pi a^2\right)}{4\pi\,\mu_o\,r^3}\left[2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right]$$

Electric and Magnetic Dipoles and Dipole Moments





$$\vec{E}(|\vec{r}| >> d) = -\nabla\phi(\vec{r})$$

$$\approx \frac{qd}{4\pi\varepsilon_o r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right)$$

Electric dipole moment:

$$\boxed{\vec{p} = q\,\vec{d}}$$

$$\vec{H}(|\vec{r}| >> a) = \frac{\nabla\times\vec{A}}{\mu_o}$$

$$\approx \frac{\mu_o I\left(\pi a^2\right)}{4\pi\,\mu_o\,r^3}\left[2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right]$$

Magnetic dipole moment:

$$\boxed{\vec{m} = I\left(\pi a^2\right)\hat{n}}$$

ECE 303 – Fall 2007 – Farhan Rana – Cornell University

Magnetic Flux and Vector Potential Line Integral

The magnetic flux λ through a surface is the surface integral of the B-field through the surface

$$\lambda = \iint \vec{B} \, . d\vec{a}$$
$$= \mu_o \iint \vec{H} \, . d\vec{a}$$

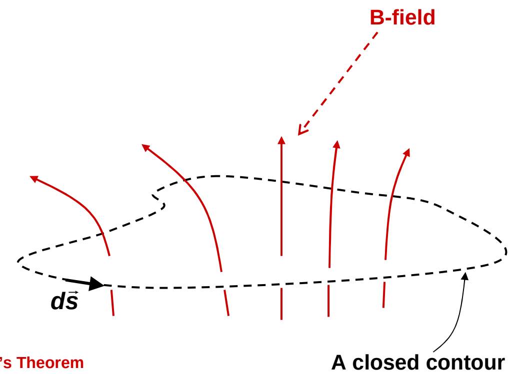
Since:

$$\vec{B} = \mu_o \vec{H} = \nabla \times \vec{A}$$

We get:

$$\lambda = \iint \vec{B} \, . d\vec{a}$$
$$= \iint \left(\nabla \times \vec{A} \right) . d\vec{a}$$
$$= \oint \vec{A} \, . \, d\vec{s}$$

Stoke's Theorem



The magnetic flux through a surface is equal to the line-integral of the vector potential along a closed contour bounding that surface

ECE 303 – Fall 2007 – Farhan Rana – Cornell University