Lecture 4

Electric Potential

In this lecture you will learn:

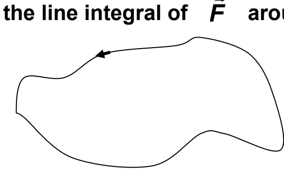
- Electric Scalar Potential
- Laplace's and Poisson's Equation
- Potential of Some Simple Charge Distributions

Conservative or Irrotational Fields

Irrotational or Conservative Fields:

Vector fields $\vec{F}$ for which $\nabla \times \vec{F} = 0$ are called "irrotational" or "conservative" fields

- This implies that the line integral of $\vec{F}$ around any closed loop is zero



$$\oint \vec{F} . d\vec{s} = 0$$

Equations of Electrostatics:

Recall the equations of electrostatics from a previous lecture:

$$\nabla . \varepsilon_o \vec{E} = \rho$$

$$\nabla \times \vec{E} = 0$$

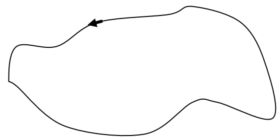
$\Rightarrow$ In electrostatics or electroquasistatics, the E-field is conservative or irrotational

(But this is not true in electrodynamics)

Conservative or Irrotational Fields

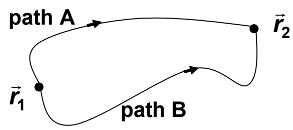
More on Irrotational or Conservative Fields:

• If the line integral of $\vec{F}$ around any closed loop is zero …..



$$\oint \vec{F} . d\vec{s} = 0$$

…. then the line integral of $\vec{F}$ between any two points is independent of any specific Path (i.e. the line integral is the same for all possible paths between the two points)



$$\oint \vec{F} . d\vec{s} = 0$$

$$\Rightarrow \left(\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} . d\vec{s} \right)_{\text{path } A} + \left(\int_{\vec{r}_2}^{\vec{r}_1} \vec{F} . d\vec{s} \right)_{\text{path } B} = 0$$

$$\Rightarrow \left(\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} . d\vec{s} \right)_{\text{path } A} - \left(\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} . d\vec{s} \right)_{\text{path } B} = 0$$

$$\Rightarrow \left(\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} . d\vec{s} \right)_{\text{path } A} = \left(\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} . d\vec{s} \right)_{\text{path } B}$$

The Electric Scalar Potential - I

The scalar potential:

Any conservative field can always be written (up to a constant) as the gradient of some scalar quantity. This holds because the curl of a gradient is always zero.

If $\vec{F} = \nabla \varphi$

Then $\nabla \times \left(\vec{F}\right) = \nabla \times (\nabla \varphi) = 0$

For the conservative E-field one writes: $\vec{E} = -\nabla \phi$
(The –ve sign is just a convention)

Where ϕ is the scalar electric potential

The scalar potential is defined only up to a constant

If the scalar potential $\phi(\vec{r})$ gives a certain electric field then the scalar potential $\phi(\vec{r}) + c$ will also give the same electric field (where c is a constant)

The absolute value of potential in a problem is generally fixed by some physical reasoning that essentially fixes the value of the constant c

The Electric Scalar Potential - II

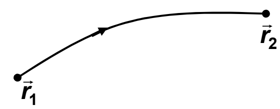
We know that:

$$\vec{E} = -\nabla\phi$$

This immediately suggests that:

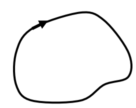
- **The line integral of E-field between any two points is the difference of the potentials at those points**

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}.d\vec{s} = \int_{\vec{r}_1}^{\vec{r}_2} (-\nabla\phi).d\vec{s} = \phi(\vec{r}_1) - \phi(\vec{r}_2)$$



- **The line integral of E-field around a closed loop is zero**

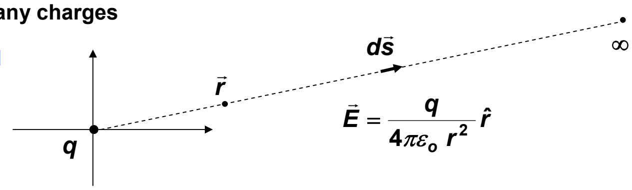
$$\oint \vec{E}.d\vec{s} = \oint (-\nabla\phi).d\vec{s} = 0$$



The Electric Scalar Potential of a Point Charge

Assumption: **The scalar potential is assumed to have a value equal to zero at infinity far away from any charges**

Point Charge Potential



$$\vec{E} = \frac{q}{4\pi\varepsilon_o r^2}\hat{r}$$

Do a line integral from infinity to the point $\vec{r}$ where the potential needs to be determined

$$\int_{\vec{r}}^{\infty} \vec{E}.d\vec{s} = \int_{\vec{r}}^{\infty} (-\nabla\phi).d\vec{s} = \phi(\vec{r}) - \cancelto{0}{\phi(\infty)} = \phi(\vec{r}) \quad \Rightarrow \quad \boxed{\phi(\vec{r}) = \int_{\vec{r}}^{\infty} \vec{E}.d\vec{s}}$$

$$\phi(\vec{r}) = \int_{\vec{r}}^{\infty} \vec{E}.d\vec{s} = \int_{r}^{\infty} \frac{q}{4\pi\varepsilon_o r^2}\,dr$$

$$= \frac{q}{4\pi\varepsilon_o r}$$

$$\boxed{\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_o r}}$$

Electric Scalar Potential and Electric Potential Energy

The electric scalar potential is the potential energy of a unit positive charge in an electric field

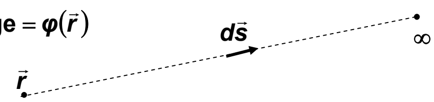
• Electric force on a charge of q Coulombs = $q\vec{E}$ (Lorentz Law)

$$\left\{ \begin{array}{l} \text{Potential energy of a charge } q \text{ at} \\ \text{any point in an electric field} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work done by the field in moving the} \\ \text{charge } q \text{ from that point to infinity} \end{array} \right\}$$

$$\text{Work done} = \int_{\vec{r}}^{\infty} \vec{F}.d\vec{s} = \int_{\vec{r}}^{\infty} q\vec{E}.d\vec{s} = q[\phi(\vec{r}) - \phi(\infty)] = q\phi(\vec{r})$$

$$\text{Work done on unit charge} = \frac{q\varphi(\vec{r})}{q} = \varphi(\vec{r})$$

$$\Rightarrow \text{P.E. of unit charge} = \varphi(\vec{r})$$



$$\Rightarrow \boxed{\text{Potential energy of a charge of } q \text{ Coulombs in electric field} = q\,\phi(\vec{r})}$$

Poisson's and Laplace's Equation

• It is not always easy to directly use Gauss' Law and solve for the electric fields
• Need an equation for the electric potential

Start from: $\nabla.\,\varepsilon_o\,\vec{E} = \rho$

Use: $\vec{E} = -\nabla\phi$

To get: $\nabla.\,\varepsilon_o\,(-\nabla\phi) = \rho$

$$\Rightarrow \boxed{\nabla^2\phi = -\frac{\rho}{\varepsilon_o}} \longrightarrow \text{Poisson's Equation}$$

If the volume charge density is zero then Poisson's equation becomes:

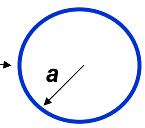
$$\boxed{\nabla^2\phi = 0} \longrightarrow \text{Laplace's Equation}$$

Poisson's or Laplace's equation can be solved to give the electric scalar potential for charge distributions

Potential of a Uniformly Charged Spherical Shell - I

Use the spherical coordinate system

σ Coulombs/m^2



For $a \le r \le \infty$ **:**

$$\nabla^2 \phi = 0$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 0$$

Assume a solution:

$$\phi(\vec{r}) = \frac{A}{r} + F$$

F must be 0 so that the potential is 0 at $r = \infty$

For $0 \le r \le a$ **:**

$$\nabla^2 \phi = 0$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 0$$

Assume solution:

$$\phi(\vec{r}) = \frac{B}{r} + D$$

Potential must not become infinite at $r = 0$ so B must be 0

Potential of a Uniformly Charged Spherical Shell - II

For $0 \le r \le a$

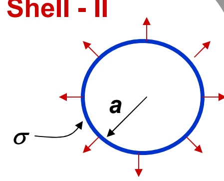
$$\phi(\vec{r}) = D$$

$$E_r(r) = -\frac{\partial \phi}{\partial r} = 0$$

For $a \le r \le \infty$

$$\phi(\vec{r}) = \frac{A}{r}$$

$$E_r(r) = -\frac{\partial \phi}{\partial r} = \frac{A}{r^2}$$



Boundary conditions

We need two additional boundary conditions to determine the two unknown coefficients A and D

(1) At $r = a$ the potential is continuous (i.e. it is the same just inside and just outside the charged sphere)

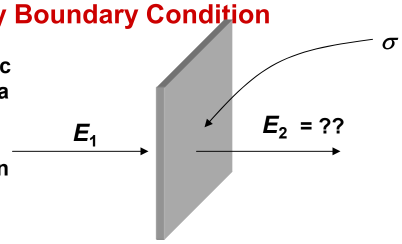
$$D = \frac{A}{a}$$

(2) At $r = a$ the electric field is NOT continuous. The jump in the component of the field normal to the shell (i.e. the radial component) is related to the surface charge density

$$\varepsilon_o \left(E_r\big|_{out} - E_r\big|_{in} \right) = \sigma$$

$$\Rightarrow \quad \varepsilon_o \left(\frac{A}{a^2} - 0 \right) = \sigma$$

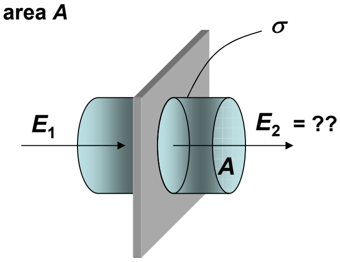
Surface Charge Density Boundary Condition



Suppose we know the surface normal electric field on just one side of a charge plane with a surface charge density σ

Question: What is the surface normal field on the other side of the charge plane?

Solution:



- Draw a Gaussian surface in the form of a cylinder of area A piercing the charge plane
- Total flux coming out of the surface = $\varepsilon_o (E_2 - E_1) A$
- Total charge enclosed by the surface = σA
- By Gauss' Law: $\varepsilon_o (E_2 - E_1) A = \sigma A$

$$\Rightarrow \quad \varepsilon_o (E_2 - E_1) = \sigma$$

$$\boxed{\varepsilon_o (E_2 - E_1) = \sigma}$$

This an extremely important result that relates surface normal electric fields on the two sides of a charge plane with surface charge density σ

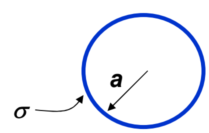
Potential of a Uniformly Charged Spherical Shell - III

For $0 \le r \le a$

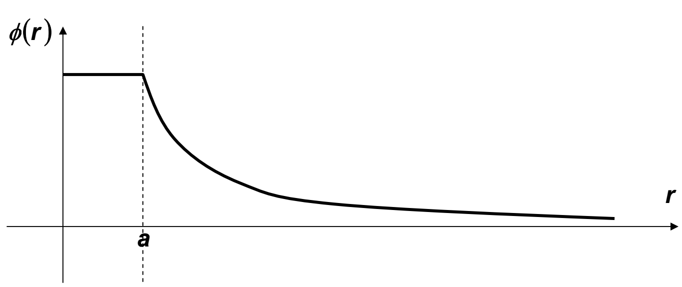
$$\phi(\vec{r}) = \frac{(4\pi\sigma a^2)}{4\pi \varepsilon_o a}$$

For $a \le r \le \infty$

$$\phi(\vec{r}) = \frac{(4\pi\sigma a^2)}{4\pi \varepsilon_o r}$$

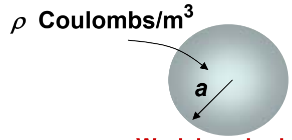


Sketch of the Potential:



Potential of a Uniformly Charged Sphere a la Poisson and Laplace

In spherical co-ordinates potential can only be a function of r (not of θ or ϕ)



Work in spherical co-ordinates

For $a \le r \le \infty$:

$$\left.\begin{aligned} &\nabla^2 \phi = 0 \\ &\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 0 \end{aligned}\right\}$$

Assume a solution:

$$\phi(\vec{r}) = \frac{A}{r} + F$$

F must be 0 so that the potential is 0 at $r = \infty$

For $0 \le r \le a$:

$$\left.\begin{aligned} &\nabla^2 \phi = -\frac{\rho}{\varepsilon_o} \\ &\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = -\frac{\rho}{\varepsilon_o} \end{aligned}\right\}$$

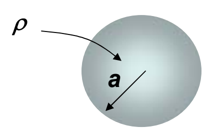
Assume solution:

$$\phi(\vec{r}) = \underbrace{\frac{B}{r} + D}_{\text{homogenous parts}} + \underbrace{C\,r^2}_{\text{particular solution}}$$

By substituting the solution in the Poisson equation find C $\longrightarrow$ $C = -\dfrac{\rho}{6\,\varepsilon_o}$

- Potential must not become infinite at $r = 0$ so B must be 0

Potential of a Uniformly Charged Sphere a la Poisson and Laplace



For $0 \le r \le a$	For $a \le r \le \infty$
$\phi(\vec{r}) = D - \dfrac{\rho}{6\,\varepsilon_o} r^2$	$\phi(\vec{r}) = \dfrac{A}{r}$

Boundary conditions

We need two additional boundary conditions to determine the two unknown coefficients A and D

(1) At $r = a$ the potential is continuous (i.e. it is the same just inside and just outside the charged sphere)

(2) At $r = a$ the radial electric field is continuous (i.e. it is the same just inside and just outside the charged sphere) $\longrightarrow$ $E_r = -\dfrac{\partial \phi}{\partial r}$

(1) gives:

$$D - \frac{\rho}{6\,\varepsilon_o} a^2 = \frac{A}{a}$$

(2) gives:

$$\frac{\rho}{3\,\varepsilon_o} a = \frac{A}{a^2}$$

$$\Rightarrow \quad A = \frac{\rho}{3\,\varepsilon_o} a^3 \qquad D = \frac{\rho}{2\,\varepsilon_o} a^2$$

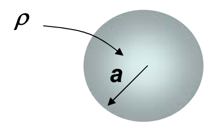
Potential of a Uniformly Charged Sphere a la Poisson and Laplace

For $0 \leq r \leq a$

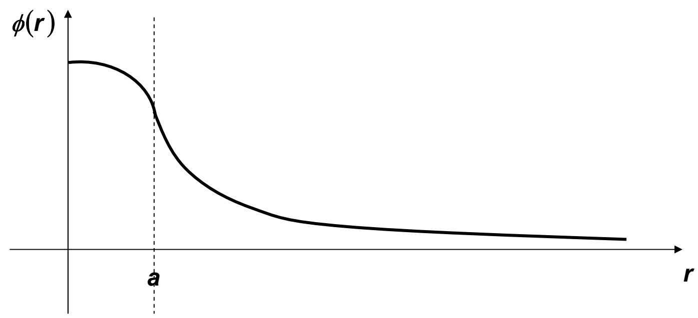
$$\phi(\bar{r}) = \frac{\rho}{2\,\varepsilon_o}\left(a^2 - \frac{r^2}{3}\right)$$

For $a \leq r \leq \infty$

$$\phi(\bar{r}) = \frac{\left(\rho \frac{4}{3}\pi a^3\right)}{4\pi\,\varepsilon_o\, r}$$



Sketch of the Potential:



The Principle of Superposition for the Electric Potential

Poisson equation is LINEAR and allows for the superposition principle to hold

- Suppose for some charge density ρ_1 one has found the potential ϕ_1
- Suppose for some other charge density ρ_2 one has found the potential ϕ_2

The superposition principle says that the sum $(\phi_1 + \phi_2)$ is the solution for the charge density $(\rho_1 + \rho_2)$

A Simple Proof

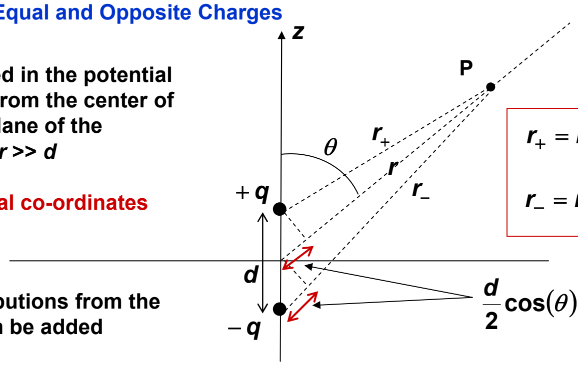
$$\nabla^2 \phi_1 = -\frac{\rho_1}{\varepsilon_o} \quad + \quad \nabla^2 \phi_2 = -\frac{\rho_2}{\varepsilon_o} \quad = \quad \nabla^2(\phi_1 + \phi_2) = -\frac{(\rho_1 + \rho_2)}{\varepsilon_o}$$

Potential of a Charge Dipole

Consider Two Equal and Opposite Charges

We are interested in the potential at a distance *r* from the center of the pair in the plane of the charges, where $r >> d$

Work in spherical co-ordinates



$$r_+ = r - \frac{d}{2}\cos(\theta)$$

$$r_- = r + \frac{d}{2}\cos(\theta)$$

Potential contributions from the two charges can be added algebraically

$$\phi(\vec{r}) = \frac{q}{4\pi\,\varepsilon_o\, r_+} - \frac{q}{4\pi\,\varepsilon_o\, r_-}$$

$$= \frac{q}{4\pi\,\varepsilon_o\left(r - \frac{d}{2}\cos(\theta)\right)} - \frac{q}{4\pi\,\varepsilon_o\left(r + \frac{d}{2}\cos(\theta)\right)}$$

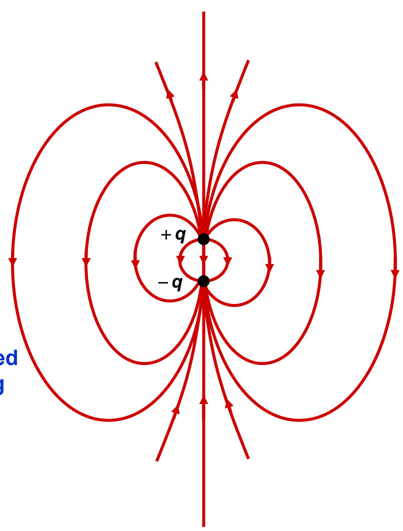
$$\approx \frac{q\,d}{4\pi\,\varepsilon_o\, r^2}\cos(\theta)$$

Field of a Charge Dipole

$$\phi(\vec{r}) \approx \frac{q\,d}{4\pi\,\varepsilon_o\, r^2}\cos(\theta)$$

$$\vec{E} = -\nabla\phi(\vec{r})$$

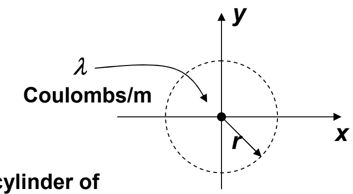
$$\approx \frac{qd}{4\pi\varepsilon_o\, r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right)$$



Same result for the E-field was obtained in the previous lecture by superposing the individual E-fields (rather than the potentials) of the two charges

Potential of a Line Charge

Consider an infinite line charge coming out of the plane of slide



Work in cylindrical co-ordinates

- The electric field, by symmetry, has only a radial component

- Draw a Gaussian surface in the form of a cylinder of radius *r* and Length *L* perpendicular to the slide

Using Gauss' Law: $\varepsilon_o E_r \left(2\pi\, r\, L\right) = \lambda\, L$

$$\Rightarrow \quad E_r = \frac{\lambda}{2\pi\,\varepsilon_o\, r}$$

But $E_r = -\nabla\phi = -\dfrac{\partial\phi}{\partial r}$ $\qquad \Rightarrow \quad \dfrac{\partial\,\phi(\vec{r})}{\partial r} = -\dfrac{\lambda}{2\pi\,\varepsilon_o\, r}$

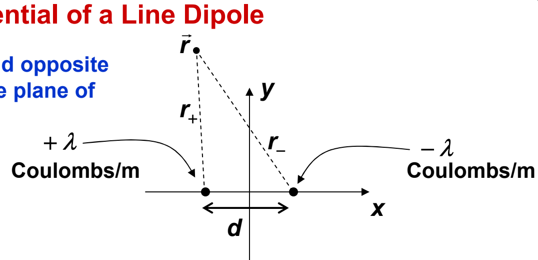
Upon integrating from r_o to r we get: $\phi(r) - \phi(r_o) = \dfrac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\dfrac{r_o}{r}\right)$

Where r_o is a constant of integration and is some point where the potential is known

The problem is that this solution becomes infinite at $r = \infty$

Potential of a Line Dipole

Consider two infinite equal and opposite line charges coming out of the plane of slide



Using superposition, the potential can be written as:

$$\phi(\vec{r}) = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_o}{r_+}\right) - \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_o}{r_-}\right)$$

$$= \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_-}{r_+}\right)$$

The final answer does not depend on the parameter r_o

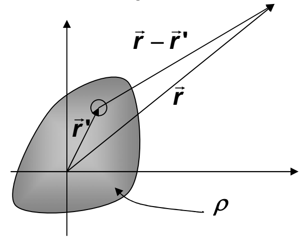
Question: where is the zero of potential?

Points for which r_+ equals r_- have zero potential. These points constitute the entire *y-z* plane

The 3D Superposition Integral for the Potential

In the most general scenario, one has to solve the Poisson equation:

$$\nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o}$$



We know that the solution for a point charge sitting at the origin:

$$\phi(\vec{r}) = \frac{q}{4\pi\,\varepsilon_o\, r}$$

To find the potential at any point one can sum up the contributions from different portions of a charge distribution treating each as a point charge

$$\phi(\vec{r}) = \iiint \frac{\rho(\vec{r}\,')}{4\pi\,\varepsilon_o\,|\vec{r} - \vec{r}\,'|}\, dV' \qquad dV' = dx'\, dy'\, dz'$$

Check: **For a point charge at the origin** $\rho(\vec{r}\,') = q\,\delta^3(\vec{r}\,') = q\,\delta(x')\delta(y')\delta(z')$

$$\phi(\vec{r}) = \iiint \frac{\rho(\vec{r}\,')}{4\pi\,\varepsilon_o\,|\vec{r} - \vec{r}\,'|}\, dV' = \iiint \frac{q\,\delta^3(\vec{r}\,')}{4\pi\,\varepsilon_o\,|\vec{r} - \vec{r}\,'|}\, dV' = \frac{q}{4\pi\,\varepsilon_o\,|\vec{r}|} = \frac{q}{4\pi\,\varepsilon_o\, r}$$

ECE 303 – Fall 2006 – Farhan Rana – Cornell University