Lecture 35

Diffraction and Aperture Antennas

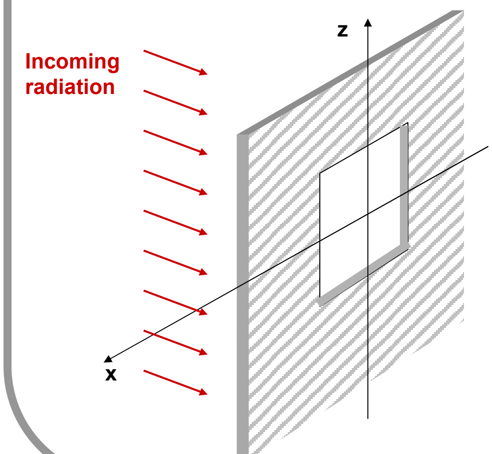
In this lecture you will learn:

- Diffraction of electromagnetic radiation
- Gain and radiation pattern of aperture antennas

Diffraction and Aperture Antennas

"Aperture antenna" usually refers to a (metallic) sheet with a hole (or an aperture) of some shape through which radiation comes out

The natural spreading of electromagnetic waves in free space when emanating from a source is called **"diffraction"**

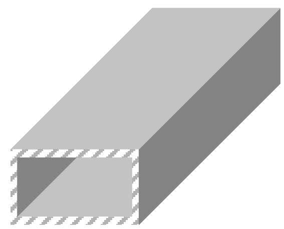


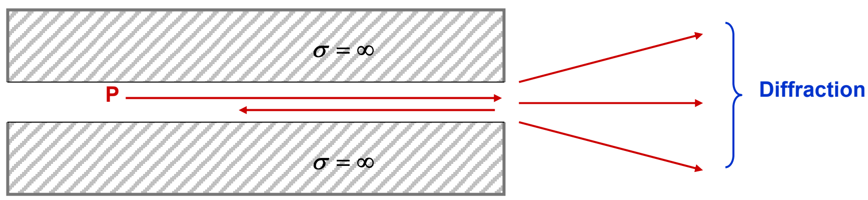
Questions

- What happens on this side?
- How does the radiation coming out of the aperture looks like when the dimensions of the hole are of the order of the wavelength?
- What is the radiation pattern?

Aperture Antennas in Practice: Rectangular Waveguides

How does radiation coming out of a rectangular waveguide looks like?

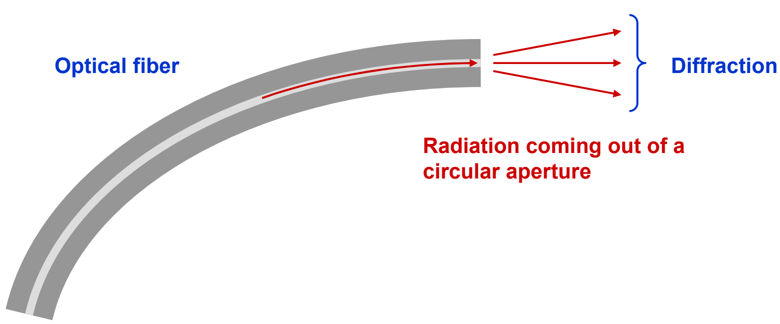




Some fraction of the incident power is reflected from the open end and some is radiated out

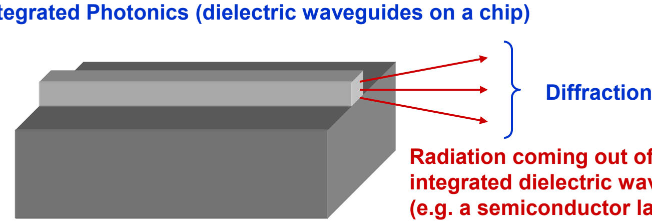
Radiation coming out of a rectangular aperture

Aperture Antennas in Practice: Dielectric Waveguides



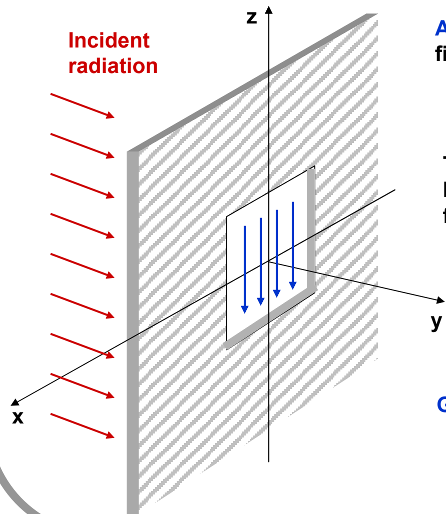
Radiation coming out of a circular aperture

Integrated Photonics (dielectric waveguides on a chip)



Radiation coming out of a integrated dielectric waveguide (e.g. a semiconductor laser)

Assumption and Goal



Assumption: Assume that we know the field for all time right at the aperture

$$\vec{E}(x, y = 0, z, t)$$

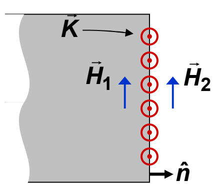
This we could know for example from our knowledge of the incident (and reflected) fields behind the aperture

Goal: To find the field for y > 0

$$\vec{E}(x, y, z, t) = ?$$

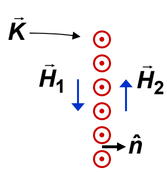
H-field and Surface Current Density Boundary Condition

First recall the surface current boundary condition for the H-field (now in vector form):



$$\hat{n} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{K}$$

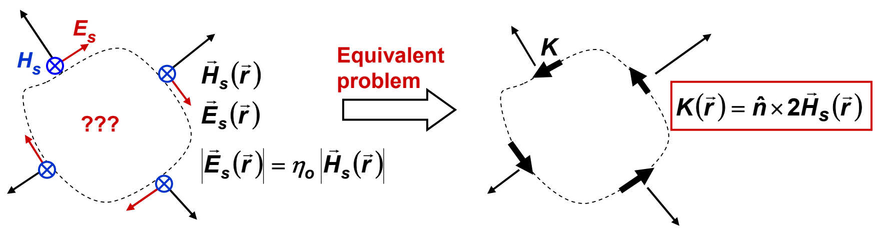
For a left-right symmetric problem:



$$\vec{H}_1 = -\vec{H}_2 \quad \Rightarrow \quad \hat{n} \times 2\vec{H}_2 = \vec{K}$$

Principle of Equivalence (Huygens Principle)

Principle of equivalence says that if one knows the radiation E- and H-fields at every point on an imaginary closed surface, then the radiation outside the closed surface can be described as the radiation generated from a surface current density that flows on the closed surface



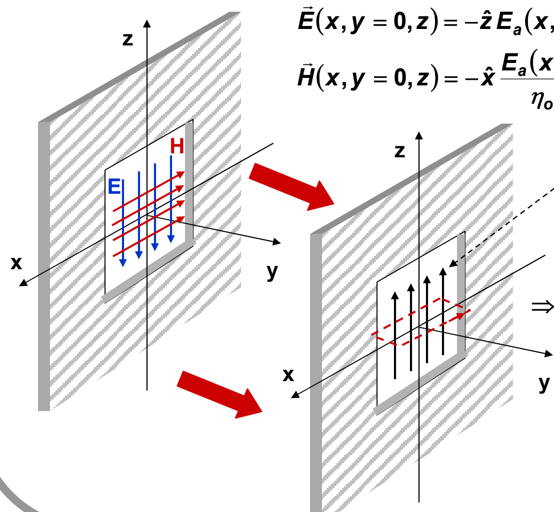
Principle of Equivalence is a mathematical statement of the old **Huygens Principle** that said that every point on a wave-front can be considered a **source of radiation**

Aperture Antenna and the Equivalent Problem

Assumption: Knowing the E-field and H-field phasors at the aperture allows us to consider the equivalent problem of radiation by a current sheet density

$$\vec{E}(x, y = 0, z) = -\hat{z}\, E_a(x,z)$$

$$\vec{H}(x, y = 0, z) = -\hat{x}\, \frac{E_a(x,z)}{\eta_o}$$

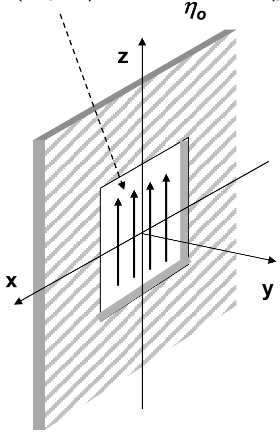


$$\vec{K}(x,z) = \hat{y} \times 2\vec{H}_a(x,z) = \hat{z}\, \frac{2E_a(x,z)}{\eta_o}$$

$$\Rightarrow \quad \vec{J}(x,y,z) = \hat{z}\, \frac{2E_a(x,z)}{\eta_o}\, \delta(y)$$

Aperture Antennas: Analysis

$$\vec{J}(x,y,z) = \hat{z}\,\frac{2\,E_a(x,z)}{\eta_o}\,\delta(y)$$



Knowing the current density, use the superposition integral for the vector potential to calculate the fields:

$$\vec{A}(\vec{r}) = \iiint \frac{\mu_o}{4\pi}\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,e^{-jk|\vec{r}-\vec{r}\,'|}\,dv'$$

Make the far-field (or the Fraunhoffer) apprximation:

$$|\vec{r}-\vec{r}\,'| \approx r - \hat{r}.\vec{r}\,'$$

$$\Rightarrow \quad \vec{A}_{ff}(\vec{r}) = \frac{\mu_o}{4\pi r}\,e^{-jkr}\iiint \vec{J}(\vec{r}\,')\;e^{jk\hat{r}.\vec{r}\,'}\,dv'$$

Compute the E-field in the far-field approximation:

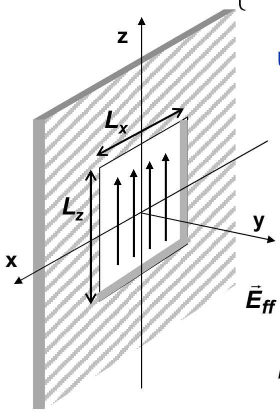
$$\vec{E}_{ff}(\vec{r}) = \frac{c^2}{j\omega}\nabla\times\nabla\times\vec{A}_{ff}(\vec{r}) \underset{\text{(in far field)}}{\approx} j\omega\,\hat{r}\times\left[\hat{r}\times\vec{A}_{ff}(\vec{r})\right]$$

$$= j\frac{\eta_o k}{4\pi r}\,e^{-jkr}\iiint \hat{r}\times\left[\hat{r}\times\vec{J}(\vec{r}\,')\right]\;e^{jk\hat{r}.\vec{r}\,'}\,dv'$$

Note that in the far-field: $\nabla\times\vec{A}_{ff}(\vec{r}) \approx -j\vec{k}\times\vec{A}_{ff}(\vec{r}) = -jk\,\hat{r}\times\vec{A}_{ff}(\vec{r})$

Rectangular Apertures: General Case

$$\vec{J}(x,y,z) = \begin{cases} \hat{z}\;2\,E_a(x,z)/\eta_o\;\delta(y) & \text{for } -\frac{L_x}{2}\le x\le\frac{L_x}{2} \;\;\&\;\; -\frac{L_z}{2}\le z\le\frac{L_z}{2} \\ 0 & \text{otherwise} \end{cases}$$



Use the formulas:

$$\vec{E}_{ff}(\vec{r}) = j\frac{\eta_o k}{4\pi r}\,e^{-jkr}\iiint \hat{r}\times\left[\hat{r}\times\vec{J}(\vec{r}\,')\right]\;e^{jk\hat{r}.\vec{r}\,'}\,dv'$$

$$\hat{r}\times\left[\hat{r}\times\hat{z}\right] = \hat{\theta}\,\sin(\theta)$$

To get:

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\,\frac{jk}{2\pi r}\sin(\theta)\,e^{-jkr}\int_{-L_z/2}^{L_z/2}\int_{-L_x/2}^{L_x/2} E_a(x',z')\;e^{j\vec{k}.\vec{r}\,'}dx'\,dz'$$

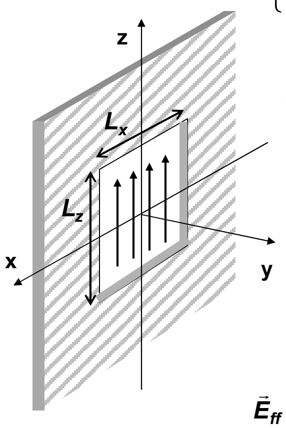
$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \qquad\qquad \vec{r}\,' = x'\,\hat{x} + z'\,\hat{z}$$

Or: $$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\,\frac{jk}{2\pi r}\sin(\theta)\,e^{-jkr}\int_{-L_z/2}^{L_z/2}\int_{-L_x/2}^{L_x/2} E_a(x',z')\;e^{jk_x x'}\,e^{jk_z z'}\,dx'\,dz'$$

Far-field is proportional to the 2D Fourier transform of the field at the aperture

Rectangular Apertures with Uniform Field at the Aperture

$$\vec{J}(x,y,z) = \begin{cases} \hat{z}\, 2E_a/\eta_o\, \delta(y) & \text{for } -\frac{L_x}{2} \le x \le \frac{L_x}{2} \;\&\; -\frac{L_z}{2} \le z \le \frac{L_z}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\,\frac{j\,k}{2\pi\, r} E_a \sin(\theta) e^{-j\,k\,r} \int_{-L_z/2}^{L_z/2}\int_{-L_x/2}^{L_x/2} e^{j\,k_x x'} e^{j\,k_z z'}\, dx'\, dz'$$

Far-field is proportional to the 2D Fourier transform of the shape of the aperture

Or:

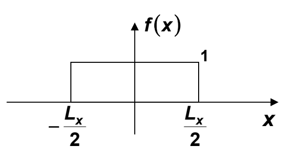
$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\,\frac{j\,k}{2\pi\, r} E_a \sin(\theta) e^{-j\,k\,r} \int_{-L_z/2}^{L_z/2} e^{j\,k_z z'}\, dz' \int_{-L_x/2}^{L_x/2} e^{j\,k_x x'}\, dx'$$

Fourier Transforms and the Rectangular Aperture Far-Field

FT: $F(k_x) = \int_{-\infty}^{\infty} f(x)\, e^{j\,k_x x}\, dx$

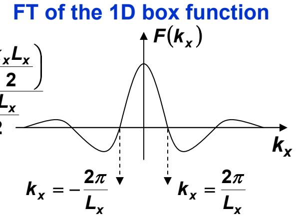
IFT: $f(x) = \int_{-\infty}^{\infty} F(k_x)\, e^{-j\,k_x x}\, \frac{dk_x}{2\pi}$

Consider the 1D box function



FT of the 1D box function

$$F(k_x) = L_x \frac{\sin\left(\frac{k_x L_x}{2}\right)}{\frac{k_x L_x}{2}}$$



Width of main lobe in k-space = $\frac{4\pi}{L_x}$

The far-field E-field is proportional to the 2D FT of the aperture shape

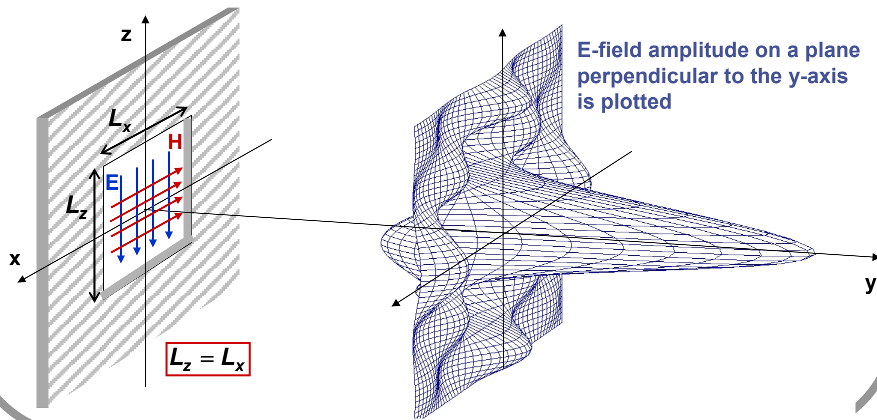
$$\vec{E}_{ff}(\vec{r}) = (\hat{\theta})\frac{j\,k}{2\pi\, r} E_a \sin(\theta) e^{-j\,k\,r} \int_{-L_x/2}^{L_x/2} e^{j\,k_x x'}\, dx' \int_{-L_z/2}^{L_z/2} e^{j\,k_z z'}\, dz'$$

$$= (\hat{\theta})\frac{j\,k}{2\pi\, r} E_a \sin(\theta) e^{-j\,k\,r} (L_x\, L_z) \frac{\sin(k_x L_x/2)}{k_x L_x/2} \frac{\sin(k_z L_z/2)}{k_z L_z/2}$$

Rectangular Aperture: Far-Field

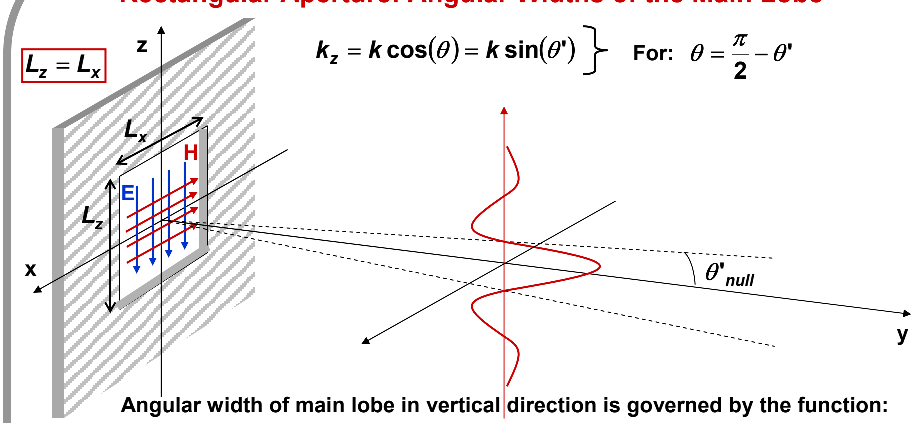
$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\,\frac{j\,k}{2\pi\,r}\,E_a\,\sin(\theta)\,e^{-jkr}\,(L_x\,L_z)\,\frac{\sin(k_x L_x/2)}{k_x L_x/2}\,\frac{\sin(k_z L_z/2)}{k_z L_z/2}$$

$$\begin{cases} k_x = k\sin(\theta)\cos(\phi) \\ k_z = k\cos(\theta) \end{cases}$$



Rectangular Aperture: Angular Widths of the Main Lobe

$$k_z = k\cos(\theta) = k\sin(\theta')\Big\} \quad \text{For: } \theta = \frac{\pi}{2} - \theta'$$



Angular width of main lobe in vertical direction is governed by the function:

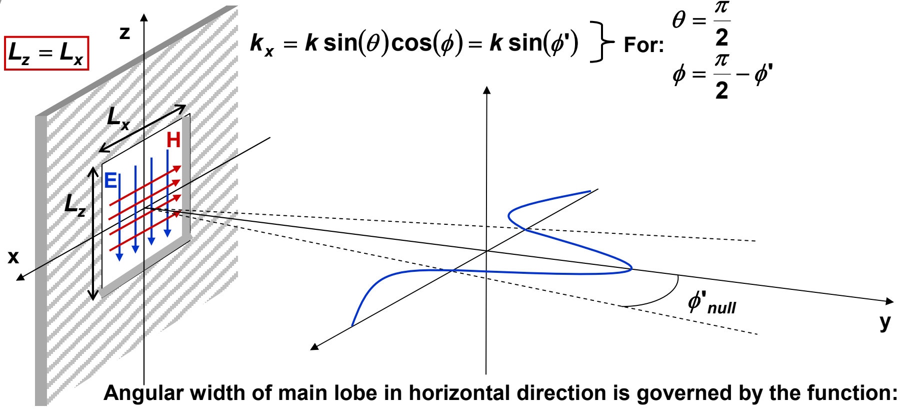
$$\frac{\sin(k_z L_z/2)}{k_z L_z/2} = \frac{\sin(k\,L_z\sin(\theta')/2)}{k\,L_z\sin(\theta')/2}$$

The angular half-width is determined by when the term inside the sine function becomes $\pm\pi$

$$\sin(\theta'_{null}) = \pm\frac{2\pi}{kL_z} = \pm\frac{\lambda}{L_z}$$

Rectangular Aperture: Angular Widths of the Main Lobe

$$k_x = k\sin(\theta)\cos(\phi) = k\sin(\phi') \quad \text{For: } \theta = \frac{\pi}{2},\ \phi = \frac{\pi}{2} - \phi'$$



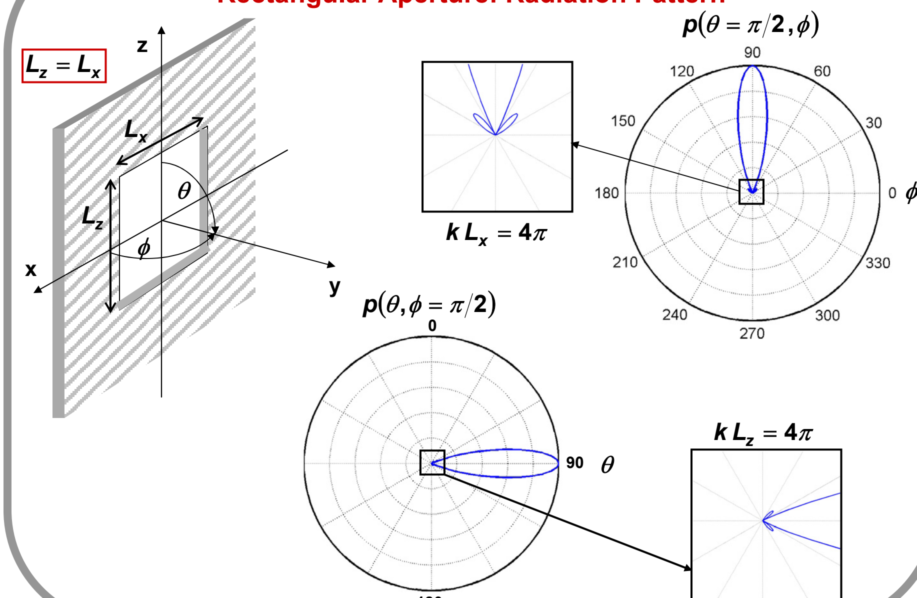
Angular width of main lobe in horizontal direction is governed by the function:

$$\frac{\sin(k_x L_x/2)}{k_x L_x/2} = \frac{\sin(k\,L_x \sin(\phi')/2)}{k\,L_x \sin(\phi')/2}$$

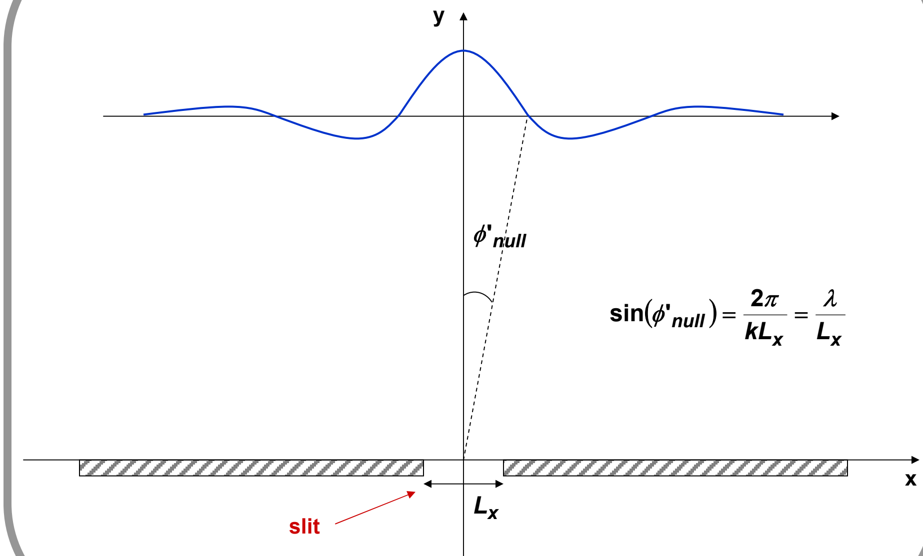
The angular half-width is determined by when the term inside the sine function becomes $\pm\pi$

$$\sin(\phi'_{null}) = \pm\frac{2\pi}{kL_x} = \pm\frac{\lambda}{L_x}$$

Rectangular Aperture: Radiation Pattern



What Causes the Nulls in the Diffraction Pattern?



$$\sin(\phi'_{null}) = \frac{2\pi}{kL_x} = \frac{\lambda}{L_x}$$

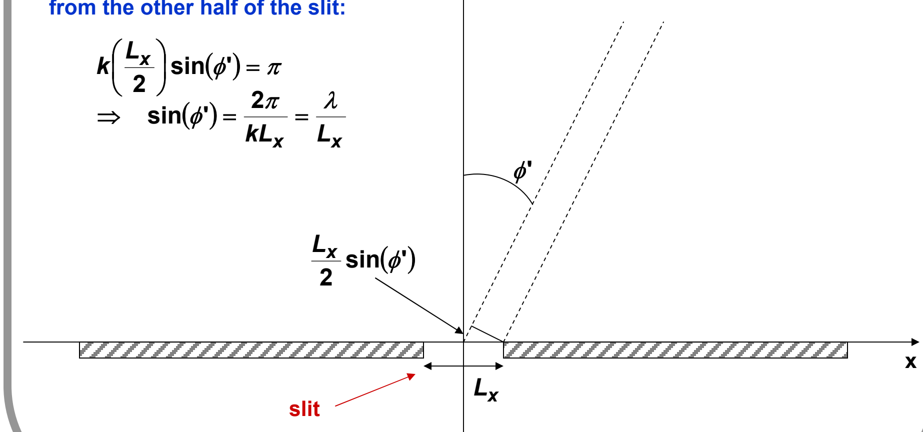
Nulls in the Diffraction Pattern: Interference in Diffraction

For the <u>first</u> null in the ϕ' direction one must have waves coming from one half of the slit interfere destructively with the waves coming from the other half of the slit:

$$k\left(\frac{L_x}{2}\right)\sin(\phi') = \pi$$

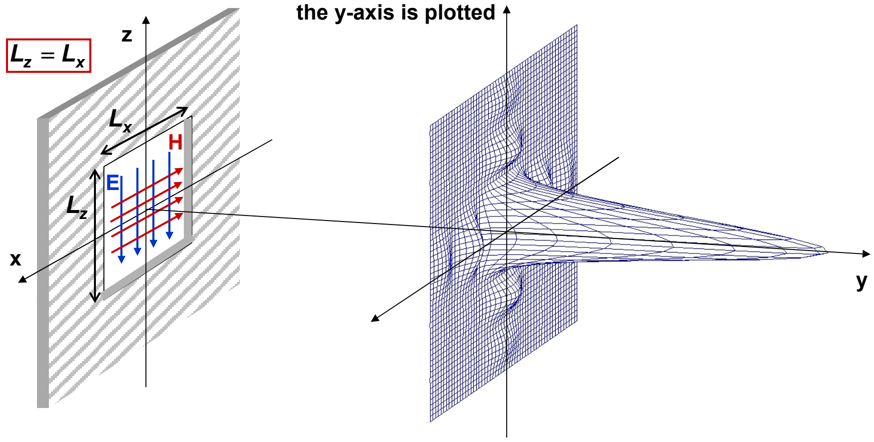
$$\Rightarrow \quad \sin(\phi') = \frac{2\pi}{kL_x} = \frac{\lambda}{L_x}$$

Can you guess what kind of interference is responsible for the <u>second</u> null? The <u>third</u> null? ……..



Rectangular Aperture: Far-Field Intensity

Intensity on a plane perpendicular to the y-axis is plotted

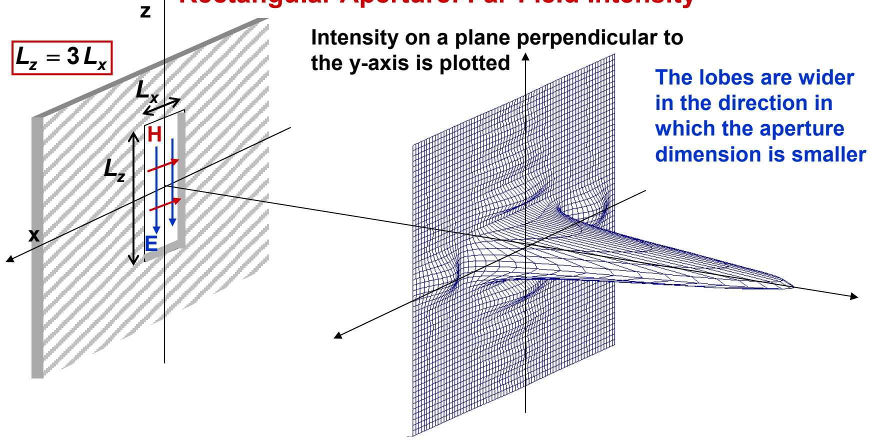


$$\langle \vec{S}_{ff}(\vec{r},t)\rangle = \hat{r}\,\frac{1}{2\eta_o}\left|\frac{k\,E_a}{2\pi\,r}\right|^2 \sin^2(\theta)\left[(L_x\ L_z)\frac{\sin(k_xL_x/2)}{k_xL_x/2}\,\frac{\sin(k_zL_z/2)}{k_zL_z/2}\right]^2$$

Rectangular Aperture: Far-Field Intensity

Intensity on a plane perpendicular to the y-axis is plotted

The lobes are wider in the direction in which the aperture dimension is smaller



$$\langle \vec{S}_{ff}(\vec{r},t)\rangle = \hat{r}\,\frac{1}{2\eta_o}\left|\frac{k\,E_a}{2\pi\,r}\right|^2 \sin^2(\theta)\left[(L_x\ L_z)\frac{\sin(k_xL_x/2)}{k_xL_x/2}\,\frac{\sin(k_zL_z/2)}{k_zL_z/2}\right]^2$$

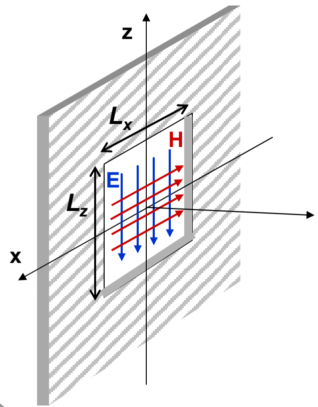
$$\Rightarrow\quad \sin(\theta'_{null}) = \pm\frac{2\pi}{kL_z} = \pm\frac{\lambda}{L_z} \qquad \sin(\phi'_{null}) = \pm\frac{2\pi}{kL_x} = \pm\frac{\lambda}{L_x}$$

Rectangular Aperture: Total Radiated Power

$$\langle \vec{S}_{ff}(\vec{r},t)\rangle = \hat{r}\,\frac{1}{2\eta_o}\left|\frac{k E_a}{2\pi r}\right|^2 \sin^2(\theta)\left[(L_x\, L_z)\,\frac{\sin(k_x L_x/2)}{k_x L_x/2}\,\frac{\sin(k_z L_z/2)}{k_z L_z/2}\right]^2$$

$$k_x = k\sin(\theta)\cos(\phi)$$

$$k_z = k\cos(\theta)$$



Total power radiated:

Calculate right at the aperture

$$P_{rad} = \int_0^{\pi}\int_0^{\pi} \langle \vec{S}_{ff}(\vec{r},t)\rangle.\hat{r}\;\; r^2\sin(\theta)\,d\theta\, d\phi$$

$$= \frac{1}{2\eta_o}|E_a|^2\, L_x\, L_z$$

Aperture Antennas: Gain and Effective Area

Gain: $G(\theta,\phi) = \eta_{rad}\,\dfrac{\langle \vec{S}_{ff}(\vec{r},t)\rangle.\hat{r}}{P_{rad}/4\pi r^2}$

$$= \frac{4\pi}{\lambda^2}\,\eta_{rad}\, L_x\, L_z\,\sin^2(\theta)\left[\frac{\sin(k_x L_x/2)}{k_x L_x/2}\,\frac{\sin(k_z L_z/2)}{k_z L_z/2}\right]^2$$

$$= \frac{4\pi}{\lambda^2}\,A(\theta,\phi)$$

Effective Area: $A(\theta,\phi) = \eta_{rad}\, L_x\, L_z\,\sin^2(\theta)\left[\dfrac{\sin(k_x L_x/2)}{k_x L_x/2}\,\dfrac{\sin(k_z L_z/2)}{k_z L_z/2}\right]^2$

Maximum Effective Area: $A(\theta,\phi)\big|_{max} = A\left(\theta = \dfrac{\pi}{2}, \phi = \dfrac{\pi}{2}\right)$

$$= \eta_{rad}\, L_x\, L_z = \eta_{rad}\,\{\text{aperture area}\}$$

Maximum possible effective area of any aperture antenna (of any shape) is equal to its actual physical area