Lecture 30

An Array of Two Hertzian Dipole Antennas

In this lecture you will learn:

- Hertzian dipole antenna arrays
- Interference and far-field radiation patterns

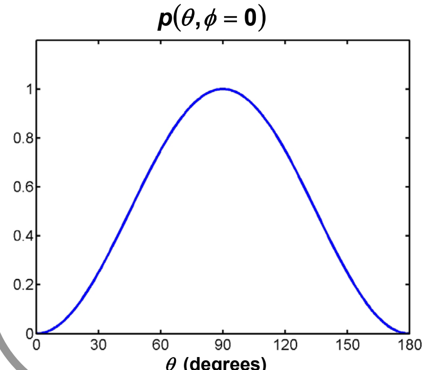
Characteristics of a Single Hertzian Dipole Antenna

Antenna Gain:

For a Hertzian dipole the gain is:

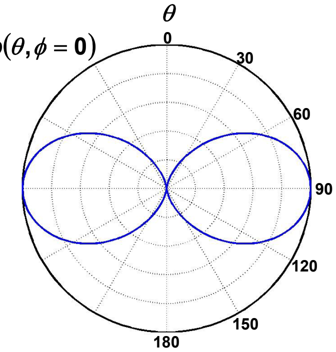
$$G(\theta,\phi)=\frac{\langle\vec{S}(\vec{r},t)\rangle.\hat{r}}{P_{rad}\big/4\pi r^2}=\frac{3}{2}\sin^2(\theta)$$

$p(\theta,\phi=0)$



θ

$p(\theta,\phi=0)$



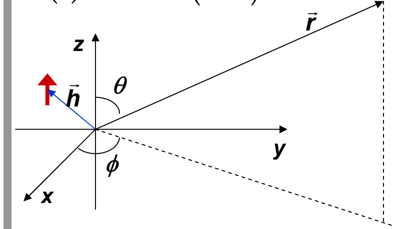
Antenna Radiation Pattern:

For a Hertzian dipole the radiation pattern is:

$$p(\theta,\phi)=\frac{G(\theta,\phi)}{G|_{max}}=\sin^2(\theta)$$

A Single Hertzian Dipole Antenna Not at Origin - I

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r} - \vec{h})$$



What if one has a Hertzian dipole sitting at some arbitrary point?

$$\vec{A}(\vec{r}) = \iiint \frac{\mu_o}{4\pi} \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} e^{-jk|\vec{r}-\vec{r}\,'|}\, dv'$$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, |\vec{r} - \vec{h}|} e^{-jk|\vec{r}-\vec{h}|}$$

If one is interested in radiation far-fields only, then assume:

$$d \ll \left\{ |\vec{h}| \;,\; \frac{\lambda}{2\pi} \right\} \ll |\vec{r}|$$

$$|\vec{r} - \vec{h}| = \sqrt{\vec{r}.\vec{r} + \vec{h}.\vec{h} - 2\,\vec{r}.\vec{h}} \approx \sqrt{\vec{r}.\vec{r} - 2\,\vec{r}.\vec{h}} = \sqrt{r^2 - 2\,\vec{r}.\vec{h}} = r\sqrt{1 - 2\frac{\vec{r}.\vec{h}}{r^2}} \approx r - \hat{r}.\vec{h}$$

So we get:

$$\vec{A}(\vec{r}) = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, |\vec{r} - \vec{h}|} e^{-jk|\vec{r}-\vec{h}|}$$

$$\approx \hat{z}\, \frac{\mu_o\, Id}{4\pi\, r} e^{-jk(r - \hat{r}.\vec{h})}$$

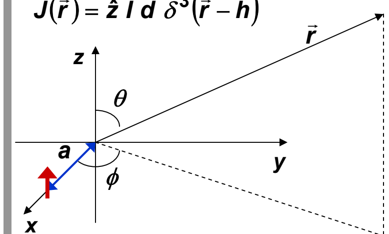
Additional phase factor

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\eta_o\, k\, Id}{4\pi\, r} \sin(\theta) e^{-jkr} \left[e^{jk\,\hat{r}.\vec{h}} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, Id}{4\pi\, r} \sin(\theta)\, e^{-jkr} \left[e^{jk\,\hat{r}.\vec{h}} \right]$$

A Single Hertzian Dipole Antenna Not at Origin - II

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r} - \vec{h})$$



Example:

Suppose: $\vec{h} = a\,\hat{x}$

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\eta_o\, k\, Id}{4\pi\, r} \sin(\theta) e^{-jkr} \left[e^{jk\,\hat{r}.\vec{h}} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, Id}{4\pi\, r} \sin(\theta)\, e^{-jkr} \left[e^{jk\,\hat{r}.\vec{h}} \right]$$

Note that:

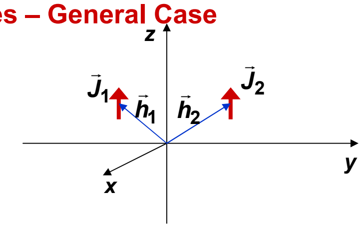
$$\hat{r} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$$

Therefore:

$$\Rightarrow \quad \vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\eta_o\, k\, Id}{4\pi\, r} \sin(\theta) e^{-jkr} \left[e^{jk\,a \sin(\theta)\cos(\phi)} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, Id}{4\pi\, r} \sin(\theta) e^{-jkr} \left[e^{jk\,a \sin(\theta)\cos(\phi)} \right]$$

Two Hertzian Dipoles – General Case



$$\vec{J}_1(\vec{r}) = \hat{z}\, I_1\, d\, \delta^3(\vec{r} - \vec{h}_1)$$

$$\vec{J}_2(\vec{r}) = \hat{z}\, I_2\, d\, \delta^3(\vec{r} - \vec{h}_2)$$

Can write the E-field and the H-field in the far-field directly:

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k\, I_1 d}{4\pi\, r} \sin(\theta) e^{-j\,k\left(r - \hat{r}.\vec{h}_1\right)} + \hat{\theta}\, \frac{j\,\eta_o\, k\, I_2 d}{4\pi\, r} \sin(\theta) e^{-j\,k\left(r - \hat{r}.\vec{h}_2\right)}$$

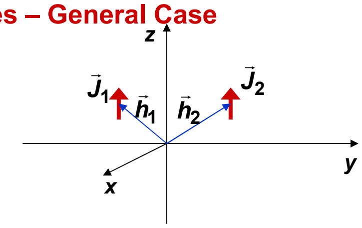
$$= \hat{\theta}\, \frac{j\,\eta_o\, I_1 k d}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r} \left[\; e^{j\,k\,\hat{r}.\vec{h}_1} + \frac{I_2}{I_1}\, e^{j\,k\,\hat{r}.\vec{h}_2} \;\right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\, k I_1 d}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r} \left[\; e^{j\,k\,\hat{r}.\vec{h}_1} + \frac{I_2}{I_1}\, e^{j\,k\,\hat{r}.\vec{h}_2} \;\right]$$

Remember that:

$$\hat{r} = \hat{x}\, \sin(\theta)\cos(\phi) + \hat{y}\, \sin(\theta)\sin(\phi) + \hat{z}\, \cos(\theta)$$

Two Hertzian Dipoles – General Case



$$\vec{J}_1(\vec{r}) = \hat{z}\, I_1\, d\, \delta^3(\vec{r} - \vec{h}_1)$$

$$\vec{J}_2(\vec{r}) = \hat{z}\, I_2\, d\, \delta^3(\vec{r} - \vec{h}_2)$$

Can write the E-field in the far-field as:

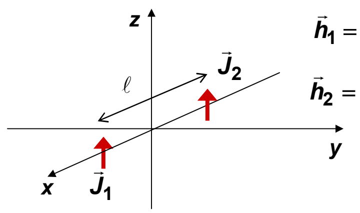
$$\vec{E}_{ff}(\vec{r}) = \underbrace{\hat{\theta}\, \frac{j\,\eta_o\, k I_1 d}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r}}_{} \underbrace{\left[\; e^{j\,k\,\hat{r}.\vec{h}_1} + \frac{I_2}{I_1}\, e^{j\,k\,\hat{r}.\vec{h}_2} \;\right]}_{}$$

Depends only on the radiating properties of the individual antennas (ELEMENT FACTOR)

Depends on the relative positions as well as the relative current amplitudes of the two antennas (ARRAY FACTOR)

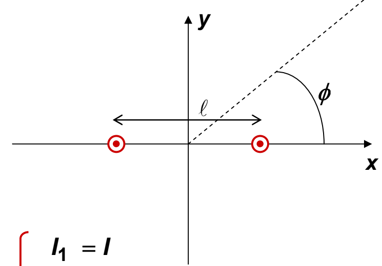
Describes INTERFERENCE in the far-field between the radiation emitted by the two dipoles

Two Hertzian Dipoles on X-Axis – Gain and Radiation Pattern - I



$$\vec{h}_1 = \hat{x}\,\frac{\ell}{2}$$

$$\vec{h}_2 = -\hat{x}\,\frac{\ell}{2}$$



The amplitudes and phases of the currents in the two dipoles are not the same:

$$\begin{cases} I_1 = I \\ I_2 = I\,A\,e^{j\alpha} \end{cases}$$

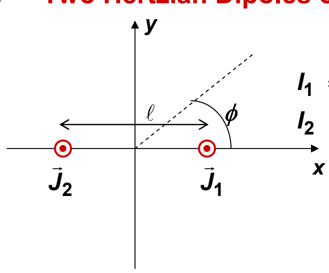
Question: What is the radiation pattern $p(\theta = \pi/2, \phi)$ in the *x-y* plane?

$$\vec{E}_{ff}(\vec{r})\Big|_{\theta=\pi/2} = \hat{\theta}\,\frac{j\,\eta_o\,k\,I\,d}{4\pi\,r}\,e^{-j\,k\,r}\left[\,e^{j\,k\,\frac{\ell}{2}\cos(\phi)} + A\,e^{j\alpha}\,e^{-j\,k\,\frac{\ell}{2}\cos(\phi)}\,\right]$$

$$\vec{H}_{ff}(\vec{r})\Big|_{\theta=\pi/2} = \hat{\phi}\,\frac{j\,k\,I\,d}{4\pi\,r}\,e^{-j\,k\,r}\left[\,e^{j\,k\,\frac{\ell}{2}\cos(\phi)} + A\,e^{j\alpha}\,e^{-j\,k\,\frac{\ell}{2}\cos(\phi)}\,\right]$$

$$\Rightarrow\quad \vec{S}_{ff}(\vec{r})\Big|_{\theta=\pi/2} = \hat{r}\,\eta_o\left|\frac{k\,I\,d}{4\,\pi\,r}\right|^2\left|\,e^{j\,k\,\frac{\ell}{2}\cos(\phi)} + A\,e^{j\alpha}\,e^{-j\,k\,\frac{\ell}{2}\cos(\phi)}\right|^2$$

Two Hertzian Dipoles on X-Axis – Gain and Radiation Pattern - II



$$I_1 = I$$

$$I_2 = I\,A\,e^{j\alpha}$$

Total Power Radiated:

$$P_{rad} = \frac{\eta_o}{12\pi}\left|k\;Id\right|^2\left(1 + A^2\right)$$

sum of the power radiated by individual dipoles

Poynting vector:

$$\vec{S}_{ff}(\vec{r})\Big|_{\theta=\pi/2} = \hat{r}\,\eta_o\left|\frac{k\,I\,d}{4\,\pi\,r}\right|^2\left|\,e^{j\,k\,\frac{\ell}{2}\cos(\phi)} + A\,e^{j\alpha}\,e^{-j\,k\,\frac{\ell}{2}\cos(\phi)}\right|^2$$

Gain:

$$G(\theta = \pi/2, \phi) = \frac{\langle \vec{S}(\vec{r},t)\rangle.\hat{r}}{P_{rad}\,/\,4\pi\,r^2} = \frac{3}{2}\left(\frac{1 + A^2 + 2A\cos[k\ell\cos(\phi) - \alpha]}{1 + A^2}\right)$$

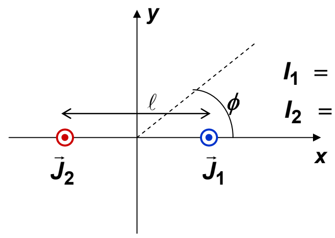
Pattern:

$$p(\theta = \pi/2, \phi) = \frac{G(\theta = \pi/2, \phi)}{G|_{\max}} = \left(\frac{1 + A^2 + 2\,A\cos[k\ell\cos(\phi) - \alpha]}{(1 + A)^2}\right)$$

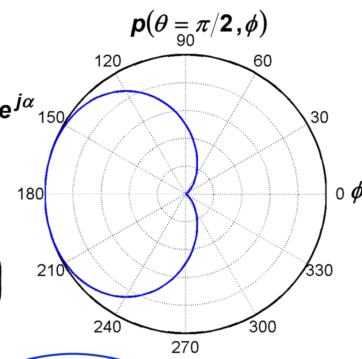
Case I: Two Hertzian Dipoles on X-Axis

Case I:

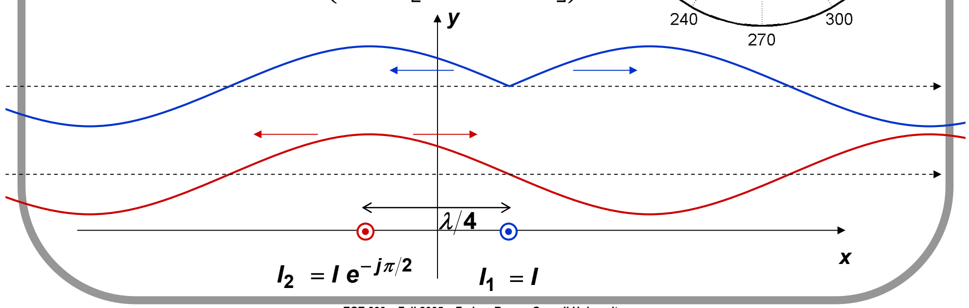
A = 1
$\ell = \lambda / 4$
$\alpha = -\pi / 2$



$I_1 = I$
$I_2 = I\,A\,e^{j\alpha}$



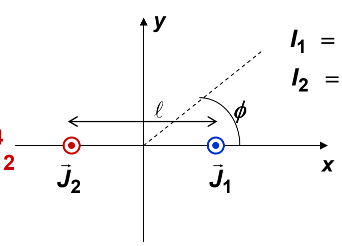
Pattern: $p(\theta = \pi/2, \phi) = \frac{1}{2}\left(1 + \cos\left[\frac{\pi}{2}\cos(\phi) + \frac{\pi}{2}\right]\right)$



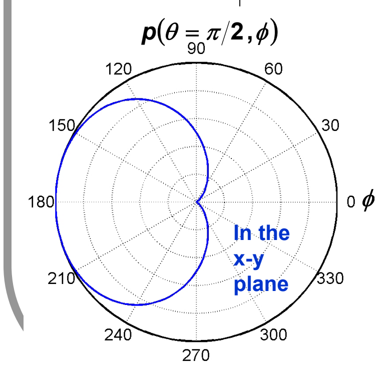
Case I: Two Hertzian Dipoles on X-Axis

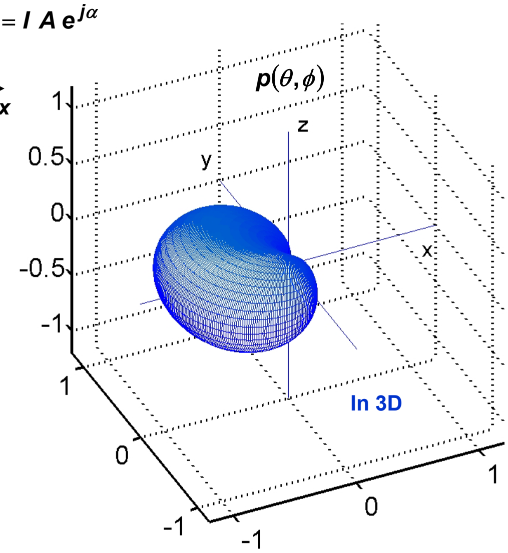
Case I:

A = 1
$\ell = \lambda / 4$
$\alpha = -\pi / 2$

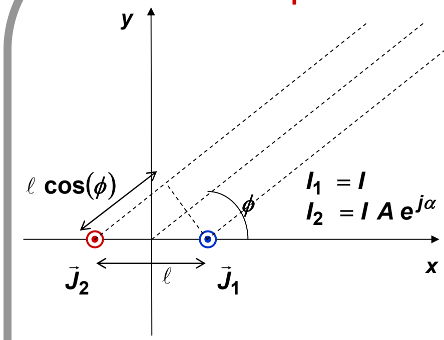


$I_1 = I$
$I_2 = I\,A\,e^{j\alpha}$





Two Hertzian Dipoles on X-Axis – Some Physical Reasoning



When traveling in the direction ϕ:

- The waves from dipole 2 have a phase lead of α compared to those from dipole 1
- But the waves from dipole 2 travel a distance $\ell \cos(\phi)$ more than from dipole 1. This means they would lag by a phase of $k\ell \cos(\phi)$

Consequently:

- The net phase difference between waves from dipole 2 and dipole 1 in the direction ϕ is:

$$\alpha - k\ell \cos(\phi)$$

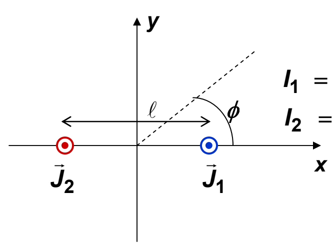
- One could therefore expect a maximum in the radiation pattern in the direction ϕ if the net phase difference is a integral multiple of 2π:

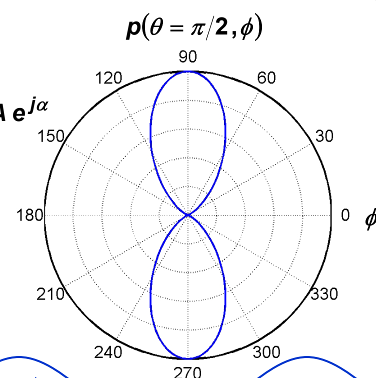
$$\alpha - k\ell \cos(\phi) = \pm 2\pi\, n \qquad \{n = 1,2,3, \ldots\ldots$$

Case II: Two Hertzian Dipoles on X-Axis

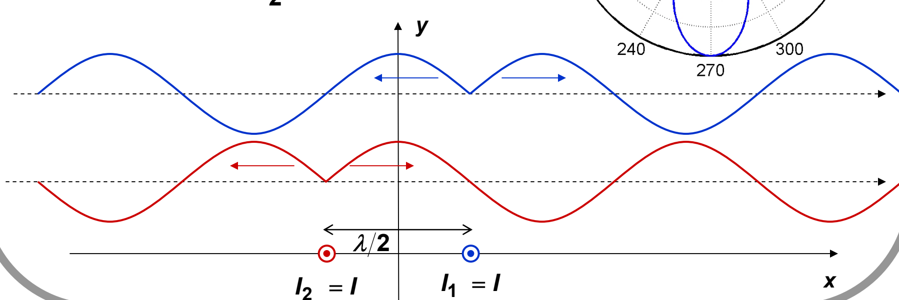
Case II:

A = 1
$\ell = \lambda / 2$
$\alpha = 0$





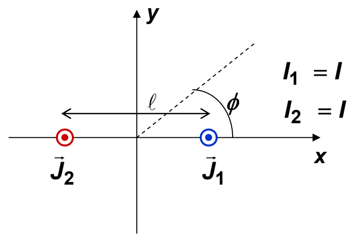
Pattern: $p(\theta = \pi/2, \phi) = \frac{1}{2}\left(1 + \cos[\pi \cos(\phi)]\right)$



Case III: Two Hertzian Dipoles on X-Axis

Case III:

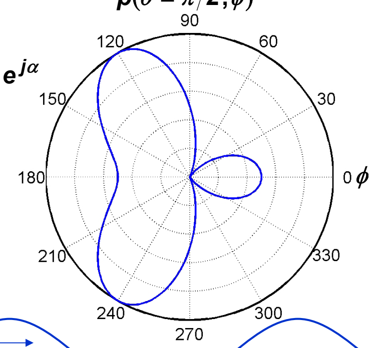
A = 1
$\ell = \lambda / 2$
$\alpha = -\pi / 2$



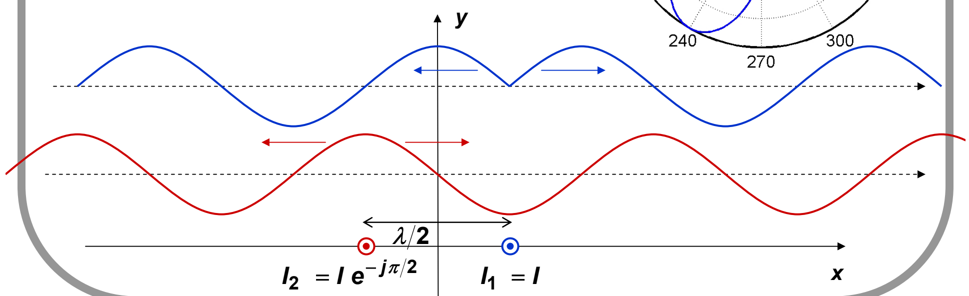
$I_1 = I$

$I_2 = I\, A\, e^{j\alpha}$

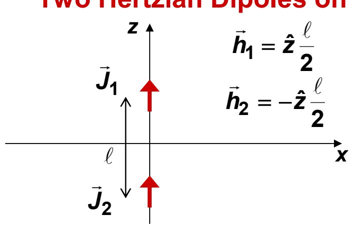
$p(\theta = \pi/2, \phi)$



Pattern: $p(\theta = \pi/2, \phi) = \frac{1}{2}\left(1 + \cos\left[\pi \cos(\phi) + \frac{\pi}{2}\right]\right)$



Two Hertzian Dipoles on Z-Axis – Gain and Radiation Pattern - I



$\vec{h}_1 = \hat{z}\frac{\ell}{2}$

$\vec{h}_2 = -\hat{z}\frac{\ell}{2}$

The amplitudes and phases of the currents in the two dipoles are not the same:

$$\begin{cases} I_1 = I \\ I_2 = I\, A\, e^{j\alpha} \end{cases}$$

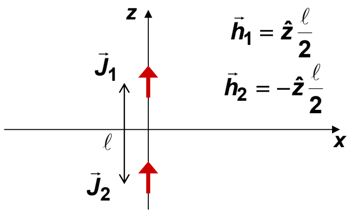
Question: What is the radiation pattern $p(\theta, \phi)$?

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k\, I\, d}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r}\left[e^{j\,k\,\frac{\ell}{2}\cos(\theta)} + A\, e^{j\alpha}\, e^{-j\,k\,\frac{\ell}{2}\cos(\theta)} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, I\, d}{4\pi\, r} \sin(\theta) e^{-j\,k\,r}\left[e^{j\,k\,\frac{\ell}{2}\cos(\theta)} + A\, e^{j\alpha}\, e^{-j\,k\,\frac{\ell}{2}\cos(\theta)} \right]$$

$$\Rightarrow \quad \vec{S}_{ff}(\vec{r}) = \hat{r}\, \eta_o \left| \frac{k\, I\, d}{4\,\pi\, r} \right|^2 \sin^2(\theta) \left| e^{j\,k\,\frac{\ell}{2}\cos(\theta)} + A\, e^{j\alpha}\, e^{-j\,k\,\frac{\ell}{2}\cos(\theta)} \right|^2$$

Two Hertzian Dipoles on Z-Axis – Gain and Radiation Pattern - II



$$\vec{h}_1 = \hat{z}\frac{\ell}{2}$$

$$\vec{h}_2 = -\hat{z}\frac{\ell}{2}$$

Total Power Radiated:

$$P_{rad} = \frac{\eta_o}{12\pi}\left|k\, Id\right|^2\left(1 + A^2\right)$$

sum of the power radiated by individual dipoles

Poynting vector:
$$\vec{S}_{ff}(\vec{r}) = \hat{r}\,\eta_o\left|\frac{k\,I\,d}{4\pi r}\right|^2 \sin^2(\theta)\left|e^{jk\frac{\ell}{2}\cos(\theta)} + A\,e^{j\alpha}\,e^{-jk\frac{\ell}{2}\cos(\theta)}\right|^2$$

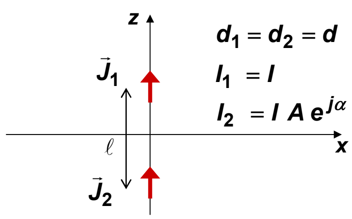
Gain:
$$G(\theta,\phi) = \frac{\langle\vec{S}(\vec{r},t)\rangle.\hat{r}}{P/4\pi r^2} = \frac{3}{2}\sin^2(\theta)\left(\frac{1 + A^2 + 2A\cos[k\ell\cos(\theta) - \alpha]}{1 + A^2}\right)$$

Pattern:
$$p(\theta,\phi) = \frac{G(\theta,\phi)}{G(\theta,\phi)|_{max}} \propto \sin^2(\theta)\left(\frac{1 + A^2 + 2A\cos[k\ell\cos(\theta) - \alpha]}{(1 + A)^2}\right)$$

Case: Two Hertzian Dipoles on Z-Axis

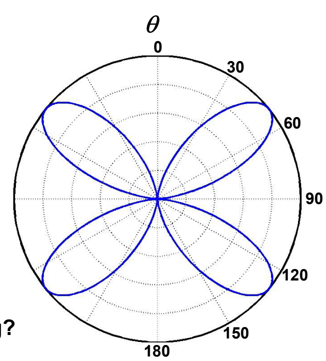
Case:

A = 1
$\ell = \lambda / 2$
$\alpha = -\pi$



$$d_1 = d_2 = d$$
$$I_1 = I$$
$$I_2 = I\,A\,e^{j\alpha}$$

$p(\theta,\phi = 0)$



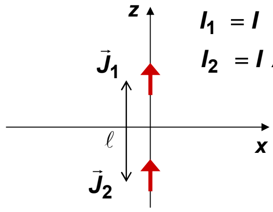
Pattern:
$$p(\theta,\phi) \propto \frac{\sin^2(\theta)}{2}\left(1 + \cos[\pi\cos(\theta) + \pi]\right)$$

Notice the locations of the nulls? Can you figure out the angular location of the nulls by physical reasoning?

Case: Two Hertzian Dipoles on Z-Axis

Case:

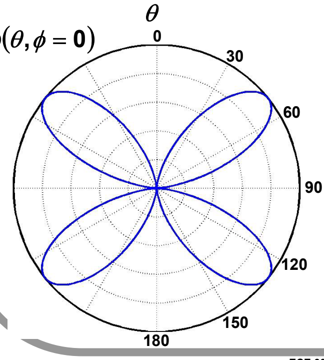
A = 1
$\ell = \lambda / 2$
$\alpha = -\pi$



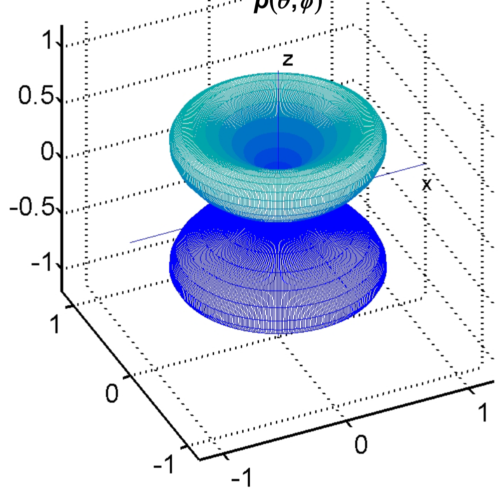
$I_1 = I$

$I_2 = I\,A\,e^{j\alpha}$

$p(\theta, \phi = 0)$

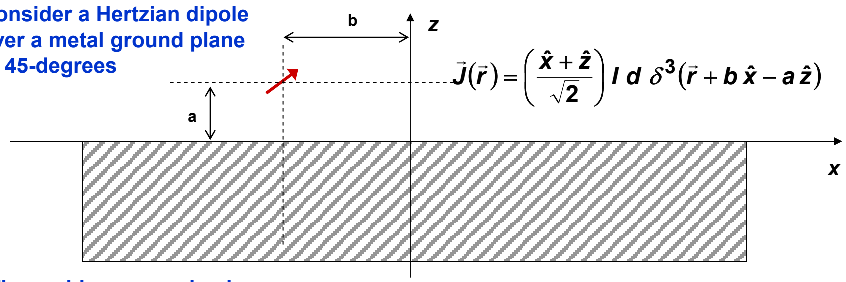


$p(\theta, \phi)$



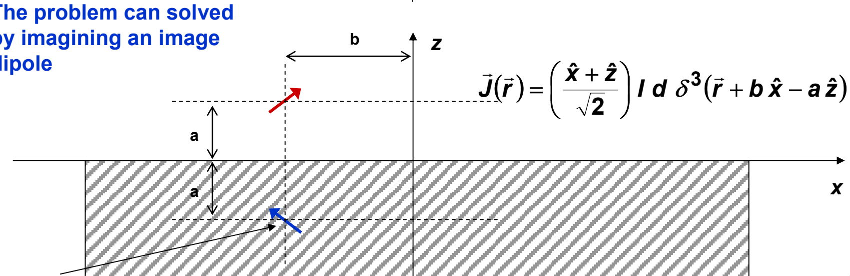
Hertzian Dipole Over a Perfect Metal Plane: Image Dipole

Consider a Hertzian dipole over a metal ground plane at 45-degrees



$$\vec{J}(\vec{r}) = \left(\frac{\hat{x}+\hat{z}}{\sqrt{2}}\right) I\, d\, \delta^3(\vec{r} + b\,\hat{x} - a\,\hat{z})$$

The problem can solved by imagining an image dipole



$$\vec{J}(\vec{r}) = \left(\frac{\hat{x}+\hat{z}}{\sqrt{2}}\right) I\, d\, \delta^3(\vec{r} + b\,\hat{x} - a\,\hat{z})$$

Image Dipole $\vec{J}(\vec{r}) = \left(\frac{-\hat{x}+\hat{z}}{\sqrt{2}}\right) I\, d\, \delta^3(\vec{r} + b\,\hat{x} + a\,\hat{z})$