Lecture 29

The Hertzian Dipole Antenna

In this lecture you will learn:

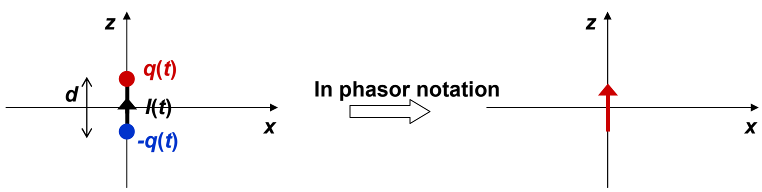
- Hertzian dipole antenna
- Gain and radiation pattern of an antenna

Hertzian Dipole Antenna - Review

- A Hertzian dipole is represented by an arrow whose direction indicates the positive direction of the current and also the orientation of the dipole in space
- By assumption, the size of the dipole is much smaller than the wavelength of the emitted radiation, i.e. $d << \lambda$

$$I(t) = I_o \cos(\omega t + \alpha)$$

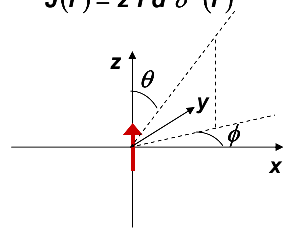
$$\vec{J}(\vec{r}) = \hat{z}\, I_o\, d\, e^{j\alpha}\, \delta^3(\vec{r})$$



- Because $d << \lambda$, the current density associated with the dipole is represented mathematically as a delta function with an appropriate weight

Radiation Emitted by a Hertzian Dipole - Review

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$$



Working in spherical coordinates

Need to solve for the radiated fields:

$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}) = -\mu_o \vec{J}(\vec{r})$$

Use the superposition integral form of the solution:

$$\vec{A}(\vec{r}) = \iiint \frac{\mu_o}{4\pi} \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} e^{-j k |\vec{r} - \vec{r}\,'|}\, dv'$$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, |\vec{r}|} e^{-j k |\vec{r}|} = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, r} e^{-j k r}$$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \left[\hat{r}\cos(\theta) - \hat{\theta}\sin(\theta)\right] \frac{\mu_o\, Id}{4\pi\, r} e^{-j k r}$$

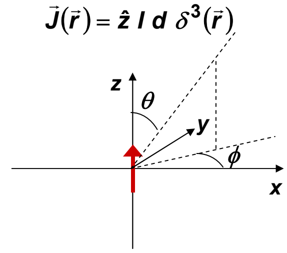
Find the H-field:

$$\mu_o \vec{H}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$$

$$\Rightarrow \quad \vec{H}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, Id}{4\pi\, r} e^{-j k r} \left[1 + \frac{1}{j k r}\right] \sin(\theta)$$

Radiation Emitted by a Hertzian Dipole - Review

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$$



Working in spherical coordinates

H-field was:

$$\vec{H}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, Id}{4\pi\, r} e^{-j k r} \left[1 + \frac{1}{j k r}\right] \sin(\theta)$$

Find the E-field:

Use Ampere's Law: $\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega\, \varepsilon_o \vec{E}(\vec{r})$

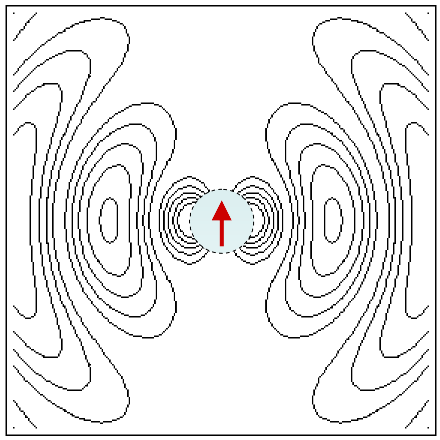
Away from the dipole the current density is zero, therefore:

$$\vec{E}(\vec{r}) = \frac{1}{j\omega\, \varepsilon_o} \nabla \times \vec{H}(\vec{r})$$

$$\vec{E}(\vec{r}) = \eta_o\, \frac{j\, k\, Id}{4\pi\, r} e^{-j k r} \left\{ \hat{r} \left[\frac{1}{j k r} + \left(\frac{1}{j k r}\right)^2\right] 2\cos(\theta) + \hat{\theta} \left[1 + \frac{1}{j k r} + \left(\frac{1}{j k r}\right)^2\right] \sin(\theta) \right\}$$

Radiation Emitted by a Hertzian Dipole - Review

$\vec{E}(\vec{r},t)$



Near field region
($r << \lambda/2\pi$)

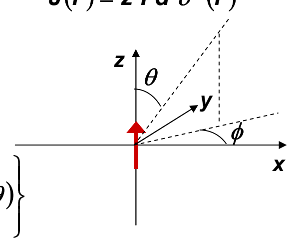
$$\vec{E}(\vec{r}) = \eta_o \frac{j\,k\,Id}{4\pi\,r} e^{-j\,k\,r} \left\{ \hat{r} \left[\frac{1}{j\,k\,r} + \left(\frac{1}{j\,k\,r} \right)^2 \right] 2\cos(\theta) + \hat{\theta} \left[1 + \frac{1}{j\,k\,r} + \left(\frac{1}{j\,k\,r} \right)^2 \right] \sin(\theta) \right\}$$

Far-Fields of a Hertzian Dipole

$$\vec{H}(\vec{r}) = \hat{\phi} \frac{j\,k\,Id}{4\pi\,r} e^{-j\,k\,r} \left[1 + \frac{1}{j\,k\,r} \right] \sin(\theta)$$

$$\vec{E}(\vec{r}) = \eta_o \frac{j\,k\,Id}{4\pi\,r} e^{-j\,k\,r} \left\{ \hat{r} \left[\frac{1}{j\,k\,r} + \left(\frac{1}{j\,k\,r} \right)^2 \right] 2\cos(\theta) + \hat{\theta} \left[1 + \frac{1}{j\,k\,r} + \left(\frac{1}{j\,k\,r} \right)^2 \right] \sin(\theta) \right\}$$

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$$



Far-field is the field far away from the dipole where: $kr >> 1$
(or more accurately where: $d << \lambda/2\pi << r$)

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi} \frac{j\,k\,Id}{4\pi\,r} e^{-j\,k\,r} \sin(\theta)$$

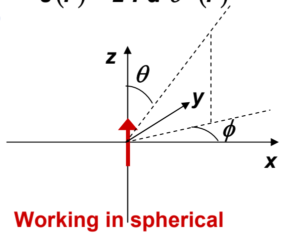
$$\vec{E}_{ff}(\vec{r}) = \hat{\theta} \frac{j\,\eta_o\,k\,Id}{4\pi\,r} e^{-j\,k\,r} \sin(\theta)$$

E-field and H-field are in phase in the far-field

Power Emitted by a Hertzian Dipole - I

$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$

The time average power per unit area going in the (θ, ϕ) direction radiated by the Hertzian dipole is given by the Poynting vector:



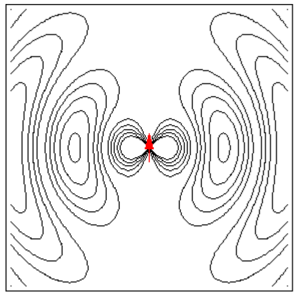
$$\langle \vec{S}(\vec{r},t) \rangle = \frac{1}{2}\mathrm{Re}\left\{ \vec{S}(\vec{r}) \right\} = \frac{1}{2}\mathrm{Re}\left\{ \vec{E}(\vec{r}) \times \vec{H}^*(\vec{r}) \right\}$$

Can simply use the field expressions in the far-field:

Working in spherical coordinates

$$= \frac{1}{2}\mathrm{Re}\left\{ \vec{E}_{ff}(\vec{r}) \times \vec{H}_{ff}^*(\vec{r}) \right\}$$

$$= \hat{r}\ \frac{\eta_o}{2}\left|\frac{k\ Id}{4\pi\ r}\right|^2 \sin^2(\theta)$$



No radiation is emitted in directions given by $\theta = 0$ or $\theta = 180$ degrees

Power Emitted by a Hertzian Dipole - II

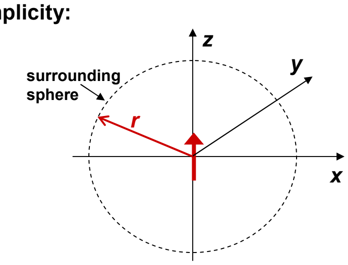
The total time average power radiated by the Hertzian dipole is given by integrating the Poynting vector over any closed surface surrounding the dipole:

Assume the closed surface to be a sphere for simplicity:

$$P_{rad} = \oiint \langle \vec{S}(\vec{r},t) \rangle . d\vec{a}$$

$$= \int_0^{2\pi}\int_0^{\pi} \langle \vec{S}(\vec{r},t) \rangle . \hat{r}\ \ r^2 \sin(\theta) d\theta\, d\phi$$

$$= \frac{\eta_o}{12\pi}\left| k\ Id \right|^2$$



Note:

For the same current, more power is radiated if d is larger, i.e. if the size of the dipole is larger.

Some Useful Characteristics of Antennas

Antenna Gain:

The gain $G(\theta,\phi)$ of an antenna is defined as the ratio of the power density (i.e. power per unit area) emitted radially outward in the (θ,ϕ) direction to the power density in the same direction radiated by an isotropic source that emits the same total power

$$G(\theta,\phi) = \frac{\langle \vec{S}(\vec{r},t)\rangle . \hat{r}}{P_{rad}/4\pi r^2} \quad \Rightarrow \quad \int_0^{2\pi}\int_0^{\pi} G(\theta,\phi)\ \sin(\theta) d\theta\, d\phi = 4\pi$$

Example: For a Hertzian dipole the gain is:

$$G(\theta,\phi) = \frac{\langle \vec{S}(\vec{r},t)\rangle . \hat{r}}{P_{rad}/4\pi r^2} = \frac{3}{2}\sin^2(\theta)$$

Antenna Radiation Pattern:

The radiation pattern $p(\theta,\phi)$ of an antenna is defined as the ratio of the gain $G(\theta,\phi)$ to the maximum value of the gain

$$p(\theta,\phi) = \frac{G(\theta,\phi)}{G|_{max}}$$

Example: For a Hertzian dipole the radiation pattern is:

$$p(\theta,\phi) = \frac{G(\theta,\phi)}{G|_{max}} = \sin^2(\theta)$$

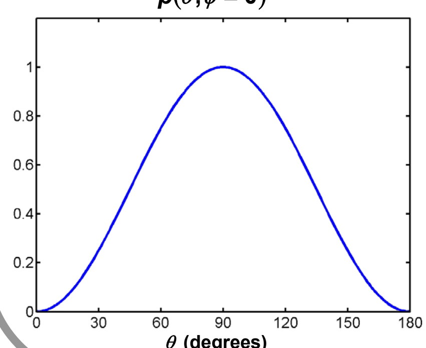
Characteristics of a Hertzian Dipole Antenna

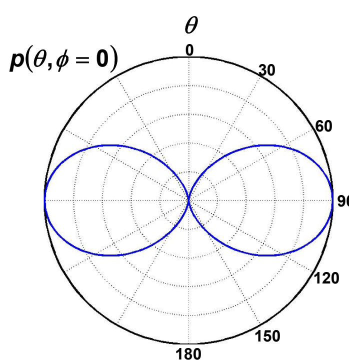
Antenna Gain:

For a Hertzian dipole the gain is:

$$G(\theta,\phi) = \frac{\langle \vec{S}(\vec{r},t)\rangle . \hat{r}}{P_{rad}/4\pi r^2} = \frac{3}{2}\sin^2(\theta)$$

$p(\theta,\phi=0)$





Antenna Radiation Pattern:

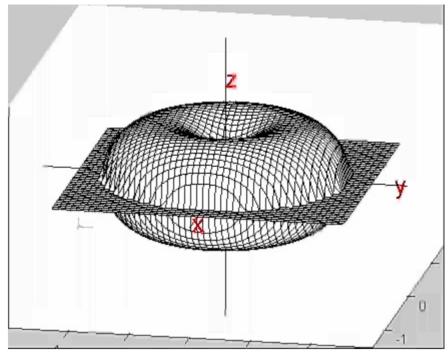
For a Hertzian dipole the radiation pattern is:

$$p(\theta,\phi) = \frac{G(\theta,\phi)}{G|_{max}} = \sin^2(\theta)$$

Characteristics of a Hertzian Dipole Antenna - II

Antenna Radiation Pattern in 3D:

$$p(\theta,\phi) = \frac{G(\theta,\phi)}{G|_{max}} = \sin^2(\theta)$$



$p(\theta,\phi)$

Radiation Resistance of Antennas

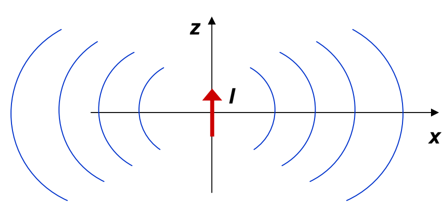
Antenna Radiation Resistance:

The radiation resistance of an antenna is found by equating the power radiated by an antenna to the power dissipated in a resistor carrying the same current as the antenna

For a Hertzian dipole:

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$$

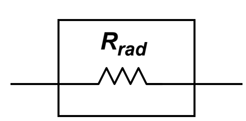
$$P_{rad} = \frac{\eta_o}{12\pi}|k\, Id|^2$$



For a Resistor:

$$P_{dissipated} = \frac{1}{2}|I|^2 R$$

Circuit Model of a Hertzian Dipole

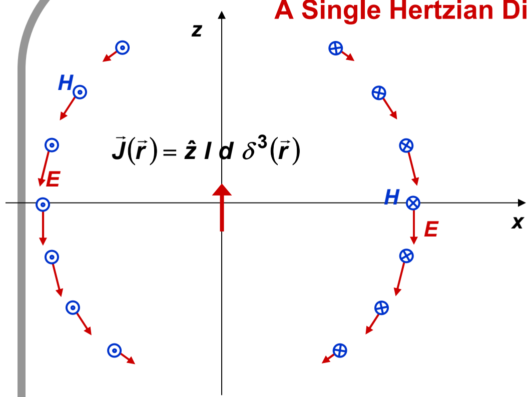


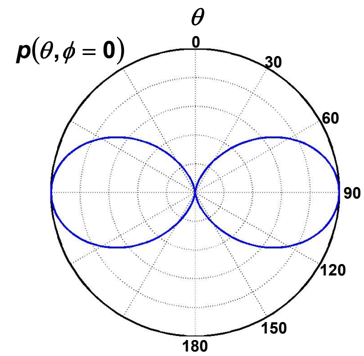
Radiation resistance of a Hertzian dipole:

$$R_{rad} = \frac{P_{rad}}{|I|^2/2} = \frac{\eta_o}{6\pi}(kd)^2$$

ECE 303 – Fall 2005 – Farhan Rana – Cornell University

A Single Hertzian Dipole Antenna





One is usually interested in only radiation far-fields:

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k\, Id}{4\pi\, r} \sin(\theta) e^{-j\,k\,r}$$

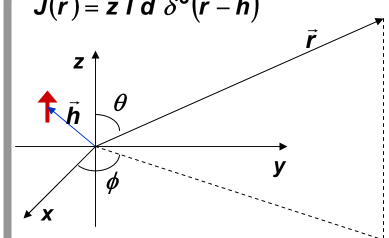
$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, Id}{4\pi\, r} \sin(\theta) e^{-j\,k\,r}$$

$$G(\theta,\phi) = \frac{\langle \vec{S}(\vec{r},t)\rangle . \hat{r}}{P / 4\pi\, r^2} = \frac{3}{2} \sin^2(\theta)$$

$$p(\theta,\phi) = \frac{G(\theta,\phi)}{G|_{max}} = \sin^2(\theta)$$

A Single Hertzian Dipole Antenna Not at Origin - I

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r} - \vec{h})$$



What if one has a Hertzian dipole sitting at some arbitrary point?

$$\vec{A}(\vec{r}) = \iiint \frac{\mu_o}{4\pi} \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} e^{-j\,k\,|\vec{r} - \vec{r}\,'|}\, dv'$$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, |\vec{r} - \vec{h}|} e^{-j\,k\,|\vec{r} - \vec{h}|}$$

If one is interested in radiation far-fields only, then assume:

$$d << \left\{ |\vec{h}| \;,\; \frac{\lambda}{2\pi} \right\} << |\vec{r}|$$

$$|\vec{r} - \vec{h}| = \sqrt{\vec{r}.\vec{r} + \vec{h}.\vec{h} - 2\,\vec{r}.\vec{h}} \approx \sqrt{\vec{r}.\vec{r} - 2\,\vec{r}.\vec{h}} = \sqrt{r^2 - 2\,\vec{r}.\vec{h}} = r\sqrt{1 - 2\frac{\vec{r}.\vec{h}}{r^2}} \approx r - \hat{r}.\vec{h}$$

So we get:

$$\vec{A}(\vec{r}) = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, |\vec{r} - \vec{h}|} e^{-j\,k\,|\vec{r} - \vec{h}|}$$

$$\approx \hat{z}\, \frac{\mu_o\, Id}{4\pi\;\, r} e^{-j\,k\,(r - \hat{r}.\vec{h})}$$

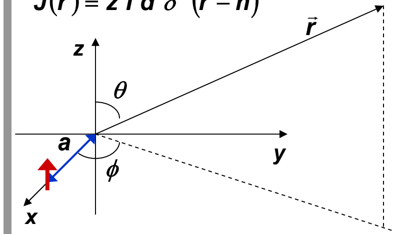
Additional phase factor

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k\, Id}{4\pi\, r} \sin(\theta) e^{-j\,k\,r} \left[e^{j\,k\;\hat{r}.\vec{h}} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\, k\, Id}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r} \left[e^{j\,k\,\hat{r}.\vec{h}} \right]$$

A Single Hertzian Dipole Antenna Not at Origin - II

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r} - \vec{h})$$



Example:

Suppose: $\vec{h} = a\,\hat{x}$

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k\, Id}{4\pi\, r} \sin(\theta) e^{-j\,k\,r} \left[e^{j\,k\,\hat{r}.\vec{h}} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\,k\,Id}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r} \left[e^{j\,k\,\hat{r}.\vec{h}} \right]$$

Note that:

$$\hat{r} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$$

Therefore:

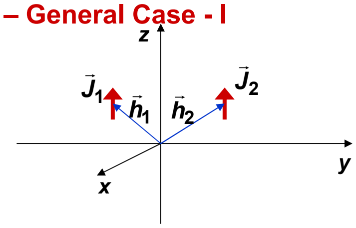
$$\Rightarrow\ \vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k\, Id}{4\pi\, r} \sin(\theta) e^{-j\,k\,r} \left[e^{j\,k\,a\sin(\theta)\cos(\phi)} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\,k\,Id}{4\pi\, r} \sin(\theta) e^{-j\,k\,r} \left[e^{j\,k\,a\sin(\theta)\cos(\phi)} \right]$$

Two Hertzian Dipoles – General Case - I

$$\vec{J}_1(\vec{r}) = \hat{z}\, I_1\, d\, \delta^3(\vec{r} - \vec{h}_1)$$

$$\vec{J}_2(\vec{r}) = \hat{z}\, I_2\, d\, \delta^3(\vec{r} - \vec{h}_2)$$



Can write the E-field and the H-field in the far-field directly:

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k\, I_1 d}{4\pi\, r} \sin(\theta) e^{-j\,k\left(r - \hat{r}.\vec{h}_1\right)} + \hat{\theta}\, \frac{j\,\eta_o\, k\, I_2 d}{4\pi\, r} \sin(\theta) e^{-j\,k\left(r - \hat{r}.\vec{h}_2\right)}$$

$$= \hat{\theta}\, \frac{j\,\eta_o\, I_1 k d}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r} \left[e^{j\,k\,\hat{r}.\vec{h}_1} + \frac{I_2}{I_1}\, e^{j\,k\,\hat{r}.\vec{h}_2} \right]$$

$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\, \frac{j\,k I_1 d}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r} \left[e^{j\,k\,\hat{r}.\vec{h}_1} + \frac{I_2}{I_1}\, e^{j\,k\,\hat{r}.\vec{h}_2} \right]$$

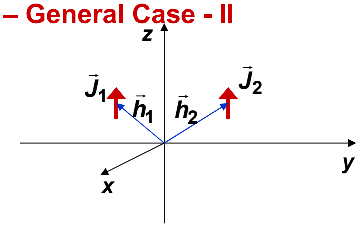
Remember that:

$$\hat{r} = \hat{x}\, \sin(\theta)\cos(\phi) + \hat{y}\, \sin(\theta)\sin(\phi) + \hat{z}\, \cos(\theta)$$

Two Hertzian Dipoles – General Case - II

$$\vec{J}_1(\vec{r}) = \hat{z}\, I_1\, d\, \delta^3\left(\vec{r} - \vec{h}_1\right)$$

$$\vec{J}_2(\vec{r}) = \hat{z}\, I_2\, d\, \delta^3\left(\vec{r} - \vec{h}_2\right)$$



Can write the E-field in the far-field as:

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\, \frac{j\,\eta_o\, k I_1 d}{4\pi\, r} \sin(\theta)\, e^{-j\,k\,r} \left[\; e^{j\,k\,\hat{r}.\vec{h}_1} + \frac{I_2}{I_1}\, e^{j\,k\,\hat{r}.\vec{h}_2} \;\right]$$

Depends only on the radiating properties of the individual antennas (ELEMENT FACTOR)

Depends on the relative positions as well as the relative current amplitudes of the two antennas (ARRAY FACTOR)

Describes INTERFERENCE in the far-field between the radiation emitted by the two dipoles