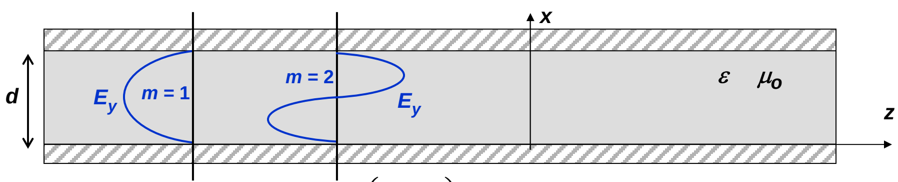
Lecture 27

Rectangular Metal Waveguides

In this lecture you will learn:

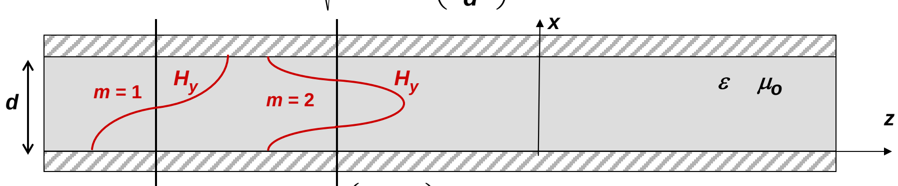
- **Rectangular metal waveguides**
- **TE and TM guided modes in rectangular metal waveguides**

Parallel Plate Metal Waveguides



TE Modes: $\vec{E}(\vec{r}) = \hat{y}\, E_o \sin\left(\frac{m\pi}{d}x\right) e^{-j k_z z}$ $\quad \{ m = 1, 2, 3, \ldots\ldots$

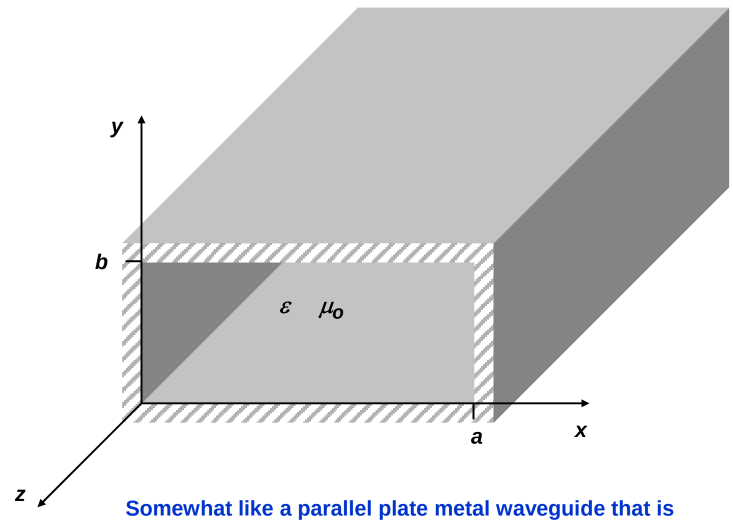
Dispersion relation: $k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{d}\right)^2}$



TM Modes: $\vec{H}(\vec{r}) = \hat{y}\, H_o \cos\left(\frac{m\pi}{d}x\right) e^{-j k_z z}$ $\quad \{ m = 0, 1, 2, 3, \ldots\ldots$

Dispersion relation: $k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{d}\right)^2}$

Rectangular Metal Waveguides



Somewhat like a parallel plate metal waveguide that is closed by metal walls on the remaining two sides

Rectangular Metal Waveguides: TE Guided Modes - I

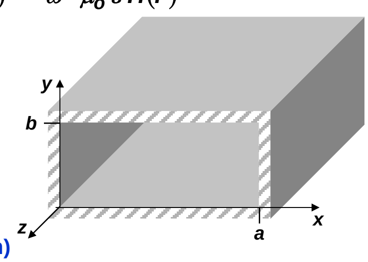
The electric field of the guided wave will satisfy the complex wave equations:

$$\left.\begin{aligned}\nabla\times\vec{E}(\vec{r}) &= -j\,\omega\,\mu_o\,\vec{H}(\vec{r})\\ \nabla\times\vec{H}(\vec{r}) &= j\,\omega\,\varepsilon\,\vec{E}(\vec{r})\end{aligned}\right\}\longrightarrow\begin{aligned}\nabla^2\vec{E}(\vec{r}) &= -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})\\ \nabla^2\vec{H}(\vec{r}) &= -\omega^2\,\mu_o\,\varepsilon\,\vec{H}(\vec{r})\end{aligned}$$

We look for solutions of the equation:

$$\nabla^2\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$



Try separation of variables (with E-field pointing in directions transverse to the direction of propagation)

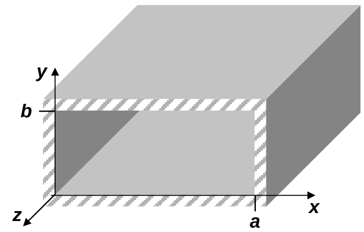
$$\vec{E}(\vec{r}) = E_o\left[\hat{y}\;A(x)\,B(y)+\hat{x}\,C(x)\,D(y)\right]e^{-j\,k_z\,z}$$

Where:

$$\left.\begin{aligned}A(x),C(x) &\propto \sin(k_x x)\ \text{ or }\ \cos(k_x x)\\ B(y),D(y) &\propto \sin(k_y y)\ \text{ or }\ \cos(k_y y)\end{aligned}\right\}\longrightarrow\boxed{k_x^2+k_y^2+k_z^2=\omega^2\,\mu_o\,\varepsilon}$$

Rectangular Metal Waveguides: TE Guided Modes - II

$$\vec{E}(\vec{r}) = E_o \, [\hat{y} \; A(x)\, B(y) + \hat{x}\, C(x)\, D(y)] e^{-j k_z z}$$



Boundary condition:

Components of E-field parallel to the metal walls must go to zero at the metal walls:

$$E_x\big|_{y=0, y=b} = 0 \qquad E_y\big|_{x=0, x=a} = 0$$

This implies:

$$A(x) = \sin(k_x x) \longrightarrow k_x = \frac{m\pi}{a} \qquad \text{where: } m = 0, 1, 2, 3, \ldots\ldots$$

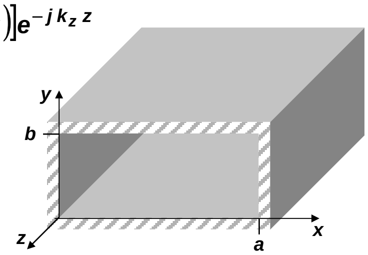
$$D(y) = \sin(k_y y) \longrightarrow k_y = \frac{n\pi}{b} \qquad \text{where: } n = 0, 1, 2, 3, \ldots\ldots$$

So we have:

$$\vec{E}(\vec{r}) = E_o \left[\hat{y} \; \sin(k_x x)\, B(y) + \hat{x}\, C(x)\, \sin(k_y y)\right] e^{-j k_z z}$$

Rectangular Metal Waveguides: TE Guided Modes - III

$$\vec{E}(\vec{r}) = E_o \left[\hat{y} \; \sin(k_x x)\, B(y) + \hat{x}\, C(x)\, \sin(k_y y)\right] e^{-j k_z z}$$



Another Boundary Condition:

Components of H-field normal to the metal walls must go to zero at the metal walls:

$$H_x\big|_{x=0, x=a} = 0 \qquad H_y\big|_{y=0, y=b} = 0$$

Turns out that the above solution form already satisfies the H-field boundary conditions

What do we do now?

We use something that we never had to use before in the context of guided waves:

Gauss' Law: $\nabla . \, \varepsilon \vec{E}(\vec{r}) = 0$

Plugging in the above solution form in Gauss' Law gives:

$$\sin(k_x x) \frac{\partial B(y)}{\partial y} = -\frac{\partial C(x)}{\partial x} \sin(k_y y)$$

Gauss' Law can only be satisfied if:

$$B(y) = \frac{k_x}{k} \cos(k_y y) \qquad C(x) = -\frac{k_y}{k} \cos(k_x x) \longrightarrow \left\{ k = \omega \sqrt{\mu_o \, \varepsilon} \right.$$

Rectangular Metal Waveguides: TE Guided Modes - IV

Finally, the solution is:

$$\vec{E}(\vec{r}) = E_o\left[\hat{y}\,\frac{k_x}{k}\sin(k_x x)\cos(k_y y) - \hat{x}\,\frac{k_y}{k}\cos(k_x x)\sin(k_y y)\right]e^{-j k_z z}$$

Where:

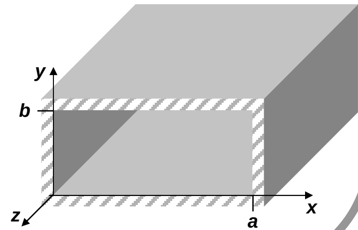
$$k_x = \frac{m\pi}{a} \qquad \text{where: } m = 0,1,2,3,\ldots\ldots$$

$$k_y = \frac{n\pi}{b} \qquad \text{where: } n = 0,1,2,3,\ldots\ldots$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu_o\varepsilon \quad \Rightarrow \quad k_z = \sqrt{\omega^2\mu_o\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

These modes are called TE_{mn} modes

By convention, the first subscript in "TE_{mn}" is related to the component of k-vector that is along the longer transverse dimension of the rectangular waveguide



NOTE: The TE_{00} mode does not exist (i.e. it corresponds to field being trivially zero everywhere)

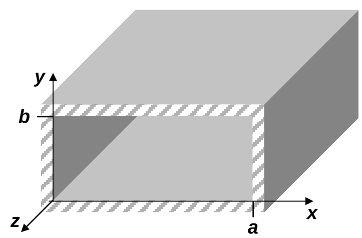
TE Guided Modes – Cut-off Frequency

$$\vec{E}(\vec{r}) = E_o\left[\hat{y}\,\frac{k_x}{k}\sin(k_x x)\cos(k_y y) - \hat{x}\,\frac{k_y}{k}\cos(k_x x)\sin(k_y y)\right]e^{-j k_z z}$$

$$k_x = \frac{m\pi}{a} \quad \{\ m = 0,1,2,3,\ldots\ldots \qquad k_y = \frac{n\pi}{b} \quad \{\ n = 0,1,2,3,\ldots\ldots$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu_o\varepsilon$$

$$\Rightarrow \quad k_z = \sqrt{\omega^2\mu_o\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$



The cut-off frequency ω_{mn} for the TE_{mn} mode is:

$$\omega_{mn} = \sqrt{\frac{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}{\mu_o\varepsilon}}$$

If the frequency ω is less than the cut-off frequency then k_z becomes entirely imaginary and the mode does not propagate (but decays exponentially with distance)

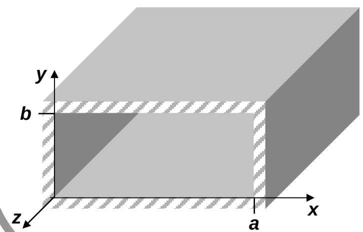
TE Guided Modes – Dispersion Relation

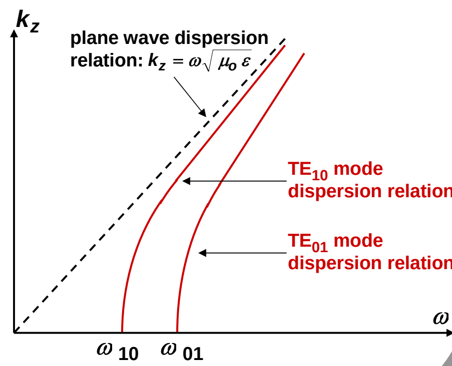
$$\vec{E}(\vec{r}) = E_o \left[\hat{y}\ \frac{k_x}{k} \sin(k_x x)\cos(k_y y) - \hat{x}\ \frac{k_y}{k} \cos(k_x x)\sin(k_y y) \right] e^{-j k_z z}$$

$$k_x = \frac{m\pi}{a} \quad \{ m = 0, 1, 2, 3, \ldots\ldots \qquad k_y = \frac{n\pi}{b} \quad \{ n = 0, 1, 2, 3, \ldots\ldots$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu_o \varepsilon$$

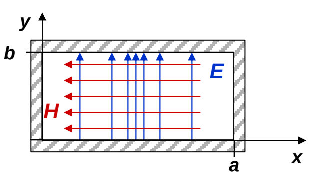
$$\Rightarrow \quad k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

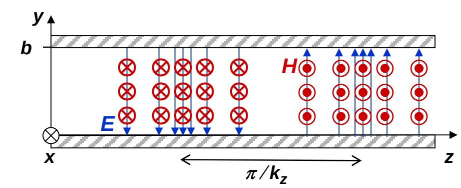


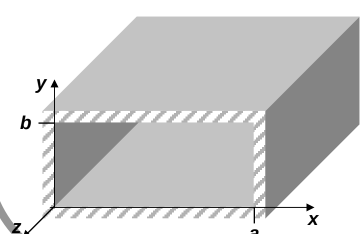


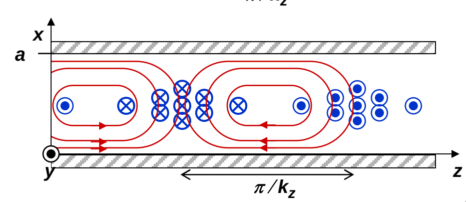
TE Guided Modes – Field Profile for TE_{10} Mode

$$\vec{E}(\vec{r}) = \hat{y}\, E_o \sin(k_x x) e^{-j k_z z} \longrightarrow \left\{ k_x = \frac{\pi}{a} \right.$$









Rectangular Metal Waveguides: TM Guided Modes - I

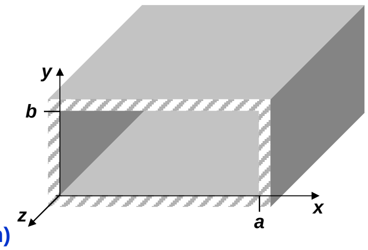
The electric field of the guided wave will satisfy the complex wave equations:

$$\left.\begin{aligned} \nabla\times\vec{E}(\vec{r}) &= -j\,\omega\,\mu_o\,\vec{H}(\vec{r}) \\ \nabla\times\vec{H}(\vec{r}) &= j\,\omega\,\varepsilon\,\vec{E}(\vec{r}) \end{aligned}\right\} \longrightarrow \begin{aligned} \nabla^2\vec{E}(\vec{r}) &= -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r}) \\ \nabla^2\vec{H}(\vec{r}) &= -\omega^2\,\mu_o\,\varepsilon\,\vec{H}(\vec{r}) \end{aligned}$$

We look for solutions of the equation:

$$\nabla^2\vec{H}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{H}(\vec{r})$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{H}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{H}(\vec{r})$$



Try separation of variables (with H-field pointing in directions transverse to the direction of propagation)

$$\vec{H}(\vec{r}) = H_o\,[\hat{y}\;A(x)\,B(y)+\hat{x}\,C(x)\,D(y)]\,e^{-j\,k_z\,z}$$

Where:

$$\left.\begin{aligned} A(x),C(x) &\propto \sin(k_x x)\;\;\text{or}\;\;\cos(k_x x) \\ B(y),D(y) &\propto \sin(k_y y)\;\;\text{or}\;\;\cos(k_y y) \end{aligned}\right\} \longrightarrow \boxed{k_x^2+k_y^2+k_z^2=\omega^2\,\mu_o\,\varepsilon}$$

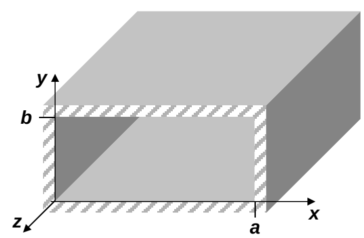
Rectangular Metal Waveguides: TM Guided Modes - II

$$\vec{H}(\vec{r}) = H_o\,[\hat{y}\;A(x)\,B(y)+\hat{x}\,C(x)\,D(y)]\,e^{-j\,k_z\,z}$$

Boundary condition:

Components of H-field normal to the metal walls must go to zero at the metal walls:

$$H_x\big|_{x=0,x=a}=0 \qquad H_y\big|_{y=0,y=b}=0$$



This implies:

$$C(x) = \sin(k_x x) \longrightarrow k_x = \frac{m\,\pi}{a} \qquad \text{where: } m = 0,1,2,3,\ldots\ldots$$

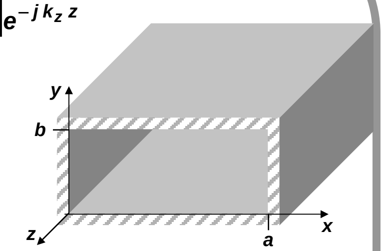
$$B(y) = \sin(k_y y) \longrightarrow k_y = \frac{n\,\pi}{b} \qquad \text{where: } n = 0,1,2,3,\ldots\ldots$$

So we have:

$$\vec{H}(\vec{r}) = H_o\left[\hat{y}\,A(x)\sin(k_y y)+\hat{x}\,\sin(k_x x)\,D(y)\right]e^{-j\,k_z\,z}$$

Rectangular Metal Waveguides: TM Guided Modes - III

$$\vec{H}(\vec{r}) = H_o\left[\hat{y}\,A(x)\sin(k_y y) + \hat{x}\,\sin(k_x x)\,D(y)\right]e^{-j k_z z}$$



Another Boundary Condition:

Components of E-field parallel to the metal walls must go to zero at the metal walls:

$$E_x\big|_{y=0,y=b} = 0 \qquad E_y\big|_{x=0,x=a} = 0$$

Turns out that the above solution form already satisfies the E-field boundary conditions

What do we do now?

Gauss' Law for H-fields: $\nabla \cdot \mu_o \vec{H}(\vec{r}) = 0$

Plugging in the above solution form in Gauss' Law for H-fields gives:

$$k_y\,A(x)\cos(k_y y) = -k_x\,D(y)\cos(k_x x)$$

Gauss' Law for H-fields can only be satisfied if:

$$A(x) = \frac{k_x}{k}\cos(k_x x) \qquad D(y) = -\frac{k_y}{k}\cos(k_y y) \longrightarrow \left\{ k = \omega\sqrt{\mu_o\,\varepsilon}\right.$$

Rectangular Metal Waveguides: TM Guided Modes - IV

Finally, the solution is:

$$\vec{H}(\vec{r}) = H_o\left[\hat{y}\,\frac{k_x}{k}\cos(k_x x)\sin(k_y y) - \hat{x}\,\frac{k_y}{k}\sin(k_x x)\cos(k_y y)\right]e^{-j k_z z}$$

Where:

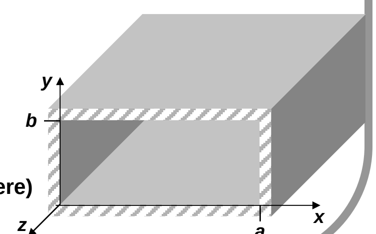
$$k_x = \frac{m\pi}{a} \qquad \text{where: } m = 0,1,2,3,\ldots\ldots$$

$$k_y = \frac{n\pi}{b} \qquad \text{where: } n = 0,1,2,3,\ldots\ldots$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu_o\,\varepsilon \;\Rightarrow\; k_z = \sqrt{\omega^2\mu_o\,\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

These modes are called TM_{mn} modes

By convention, the first subscript in "TM_{mn}" is related to the component of k-vector that is along the longer transverse dimension of the rectangular waveguide



NOTE: The TM_{00} , TM_{m0} TM_{0n} modes do not exist (i.e. they correspond to field being trivially zero everywhere)

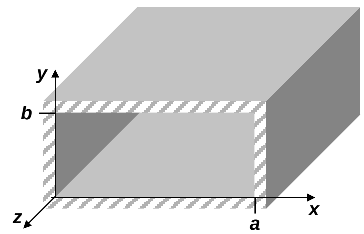
TM Guided Modes – Cut-off Frequency

$$\vec{H}(\vec{r}) = H_o \left[\hat{y}\, \frac{k_x}{k} \cos(k_x x)\, \sin(k_y y) - \hat{x}\, \frac{k_y}{k} \sin(k_x x)\, \cos(k_y y) \right] e^{-j k_z z}$$

$$k_x = \frac{m\pi}{a} \quad \{\ m = 1, 2, 3, \ldots\ldots \qquad k_y = \frac{n\pi}{b} \quad \{\ n = 1, 2, 3, \ldots\ldots$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu_o \varepsilon$$

$$\Rightarrow \quad k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$



The cut-off frequency ω_{mn} for the TM_{mn} mode is:

$$\omega_{mn} = \sqrt{\frac{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}{\mu_o \varepsilon}}$$

If the frequency ω is less than the cut-off frequency then k_z becomes entirely imaginary and the mode does not propagate (but decays exponentially with distance)

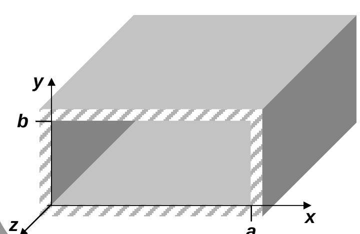
TM Guided Modes – Dispersion Relation

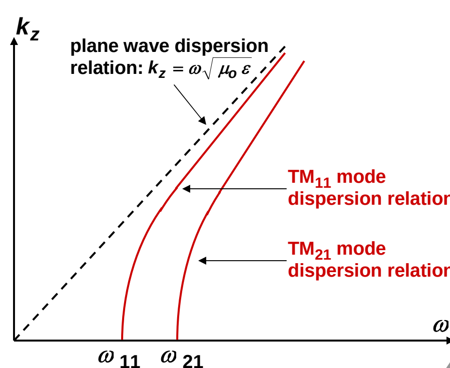
$$\vec{H}(\vec{r}) = H_o \left[\hat{y}\, \frac{k_x}{k} \cos(k_x x)\, \sin(k_y y) - \hat{x}\, \frac{k_y}{k} \sin(k_x x)\, \cos(k_y y) \right] e^{-j k_z z}$$

$$k_x = \frac{m\pi}{a} \quad \{\ m = 0, 1, 2, 3, \ldots\ldots \qquad k_y = \frac{n\pi}{b} \quad \{\ n = 0, 1, 2, 3, \ldots\ldots$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu_o \varepsilon$$

$$\Rightarrow \quad k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

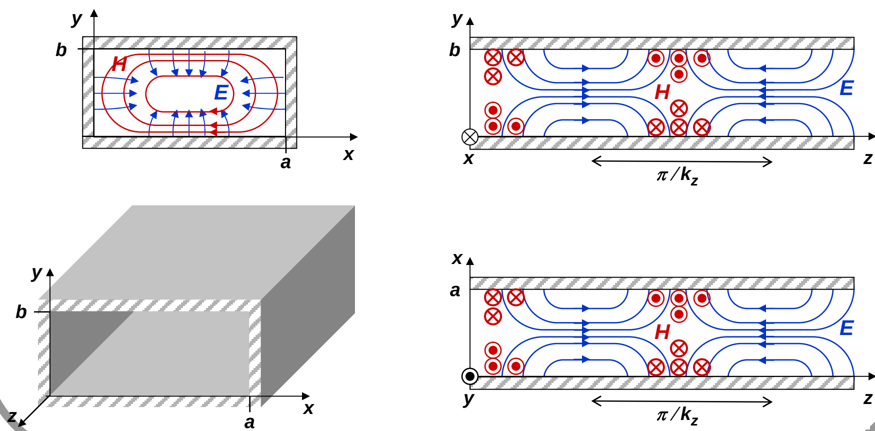




TM Guided Modes – Field Profile for TM_{11} Mode

$$\vec{H}(\vec{r}) = H_o \left[\hat{y}\, \frac{k_x}{k} \cos(k_x x)\, \sin(k_y y) - \hat{x}\, \frac{k_y}{k} \sin(k_x x)\, \cos(k_y y) \right] e^{-j k_z z}$$

$$\left\{ k_x = \frac{\pi}{a} \qquad k_y = \frac{\pi}{b} \right.$$



Rectangular Metal Waveguides

Rectangular metal waveguides are commonly used to guide electromagnetic power when dealing with high power levels (radars, satellite and space communications, wireless/mobile base stations, etc)

They are usually made of copper – and the best are gold plated

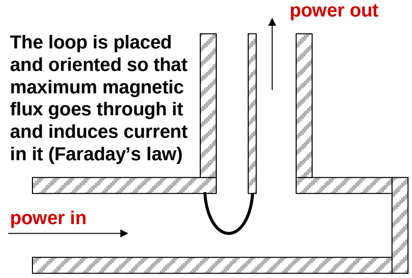
Integrated versions are used in sub-millimeter wavelength ultrahigh speed electronics (e.g. Gunn oscillators, superconducting THz electronics, nonlinear Schottky diode mixers) operating at frequencies between 300 GHz to 1000 GHz

Waveguides to Coax Adapters

Sometimes it is necessary to transfer power between a waveguide and a coax cable

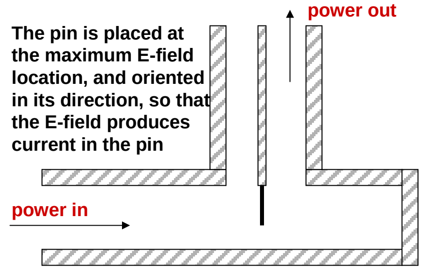
Loop Design

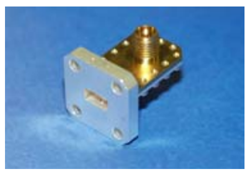
The loop is placed and oriented so that maximum magnetic flux goes through it and induces current in it (Faraday's law)



Pin Design

The pin is placed at the maximum E-field location, and oriented in its direction, so that the E-field produces current in the pin





Waveguide to Coax Adapter

Waveguide to Coax Adapters