Lecture 25

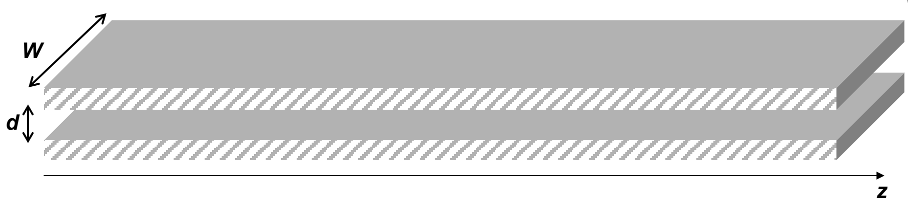
Guided Waves in Parallel Plate Metal Waveguides

In this lecture you will learn:

• Parallel plate metal waveguides

•TE and TM guided modes in waveguides

Parallel Plate Metal Waveguides



• Consider a parallel plate waveguide (shown above)

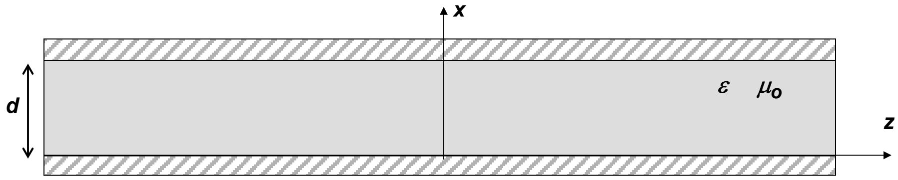
• We have studied such structures in the context of transmission lines

• We know that they can guide TEM waves (Transverse Electric and Magnetic) in which both the electric and magnetic fields point in direction perpendicular to the propagation direction

• But these structures can guide more than just the TEM waves that we have considered so far ………….

Basic Wave Equations

Consider a parallel plate waveguide:



The electric field of any guided wave will satisfy the complex wave equations:

$$\left.\begin{array}{l}\nabla\times\vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})\\ \nabla\times\vec{H}(\vec{r}) = j\,\omega\,\varepsilon\,\vec{E}(\vec{r})\end{array}\right\} \longrightarrow \begin{array}{l}\nabla^2\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})\\ \nabla^2\vec{H}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{H}(\vec{r})\end{array}$$

We look for solutions of the equation,

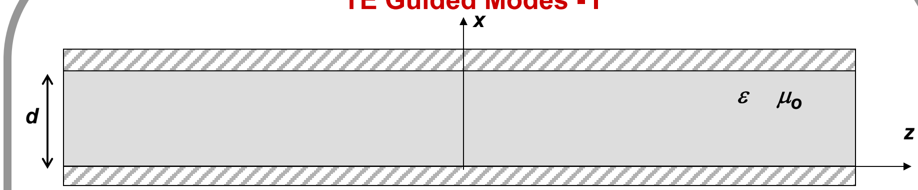
$$\nabla^2\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

where the z-dependence is that of a wave going in the z-direction, and where the E-field is pointing in the y-direction:

$$\vec{E}(\vec{r}) = \hat{y}\,F(x)\,e^{-j\,k_z\,z}$$

$F(x)$ → Some unknown function of "x"

TE Guided Modes - I



The assumed solution form: $\vec{E}(\vec{r}) = \hat{y}\,F(x)\,e^{-j\,k_z\,z}$

represents a TE guided wave (Transverse Electric) since the direction of E-field is transverse to the direction of wave propagation

Plugging the assumed solution into the equation gives:

$$\nabla^2\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

$$\Rightarrow \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

$$\Rightarrow -\frac{\partial^2 F(x)}{\partial x^2} = \left(\omega^2\,\mu_o\,\varepsilon - k_z^2\right)F(x)$$

Perfect metal boundary conditions $\Rightarrow F(x=0) = F(x=d) = 0$

TE Guided Modes - II

Need to solve: $-\dfrac{\partial^2 F(x)}{\partial x^2} = \left(\omega^2 \mu_o \varepsilon - k_z^2\right) F(x)$

With boundary conditions $\Rightarrow$ $F(x=0) = F(x=d) = 0$

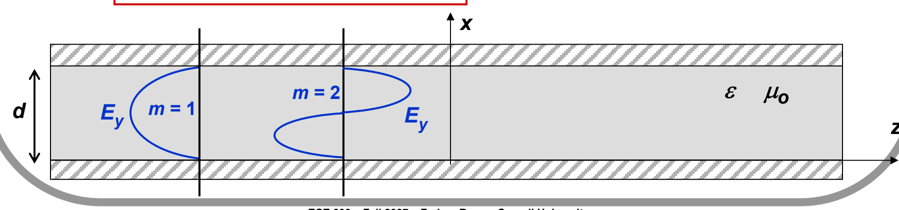
Solution is: $F(x) = E_o \sin(k_x x)$ } Automatically satisfies the boundary condition: $F(x=0)=0$

But the value of k_x cannot be arbitrary – boundary condition at $x = d$ dictates that:

$$k_x = \frac{m\pi}{d} \qquad \text{where: } m = 1, 2, 3, \ldots\ldots$$

Solution becomes: $F(x) = E_o \sin\left(\dfrac{m\pi}{d} x\right)$

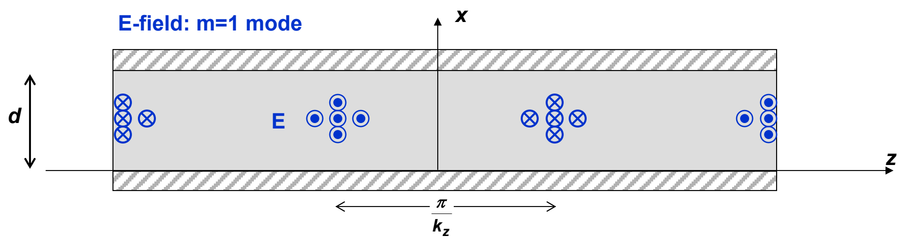
And: $\vec{E}(\vec{r}) = \hat{y}\, E_o \sin\left(\dfrac{m\pi}{d} x\right) e^{-j k_z z}$ $\{\, m = 1, 2, 3, \ldots\ldots$



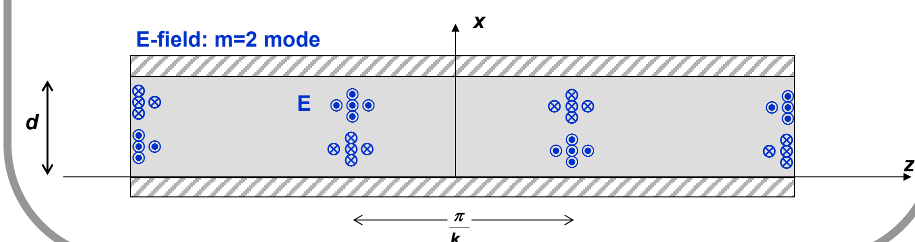
TE Guided Modes - III

$\vec{E}(\vec{r}) = \hat{y}\, E_o \sin\left(\dfrac{m\pi}{d} x\right) e^{-j k_z z}$ $\{\, m = 1, 2, 3, \ldots\ldots$

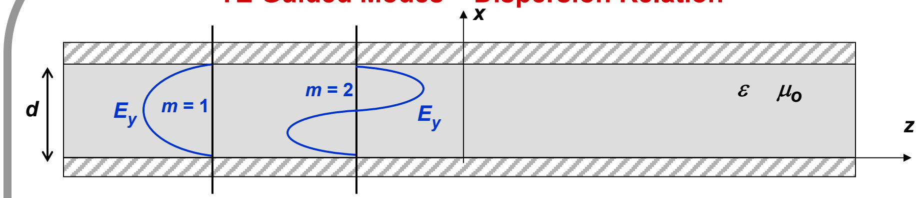
E-field: m=1 mode



E-field: m=2 mode



TE Guided Modes – Dispersion Relation



$$\vec{E}(\vec{r}) = \hat{y}\, E_o \sin\left(\frac{m\pi}{d}x\right) e^{-j k_z z} \qquad \{ m = 1, 2, 3, \ldots\ldots$$

Different "m" values correspond to different TE modes – labeled as TE_m modes

The equation: $-\frac{\partial^2 F(x)}{\partial x^2} = \left(\omega^2 \mu_o \varepsilon - k_z^2\right) F(x)$ implies:

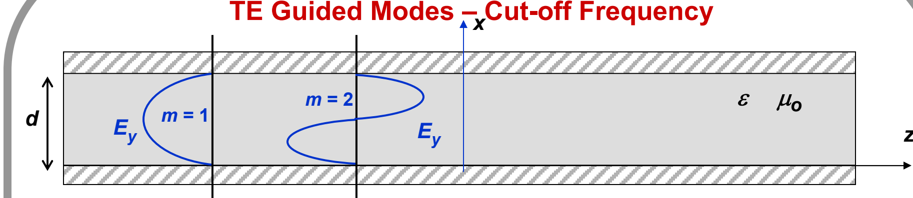
$$k_z^2 + k_x^2 = \omega^2 \mu_o \varepsilon$$

$$\Rightarrow \quad k_z^2 + \left(\frac{m\pi}{d}\right)^2 = \omega^2 \mu_o \varepsilon$$

$$\Rightarrow \quad k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{d}\right)^2}$$

} Dispersion relation for TE_m guided mode

TE Guided Modes – Cut-off Frequency



$$k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{d}\right)^2}$$

} Dispersion relation for TE_m guided mode

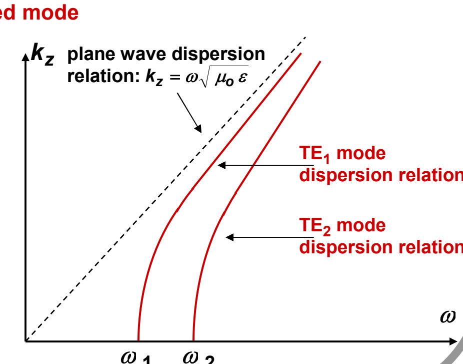
For the TE_m mode, if the frequency ω is less than:

$$\frac{1}{\sqrt{\mu_o \varepsilon}}\left(\frac{m\pi}{d}\right)$$

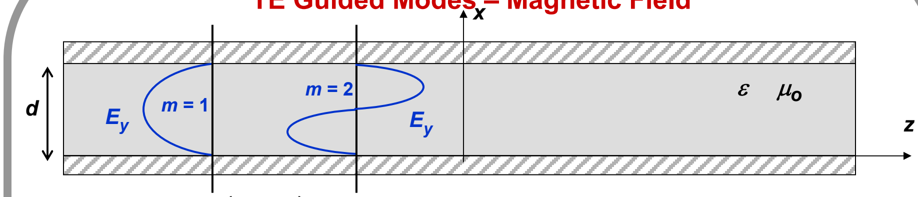
Then k_z becomes entirely imaginary and the mode does not propagate (but decays exponentially with distance)

$\Rightarrow$ Cut-off frequency for TE_m mode:

$$\omega_m = \frac{1}{\sqrt{\mu_o \varepsilon}}\left(\frac{m\pi}{d}\right)$$



TE Guided Modes – Magnetic Field



$$\vec{E}(\vec{r}) = \hat{y}\, E_o \sin\left(\frac{m\pi}{d}x\right) e^{-j k_z z} \qquad \{ m = 1, 2, 3, \ldots\ldots$$

Magnetic field is given by the equation: $\nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})$

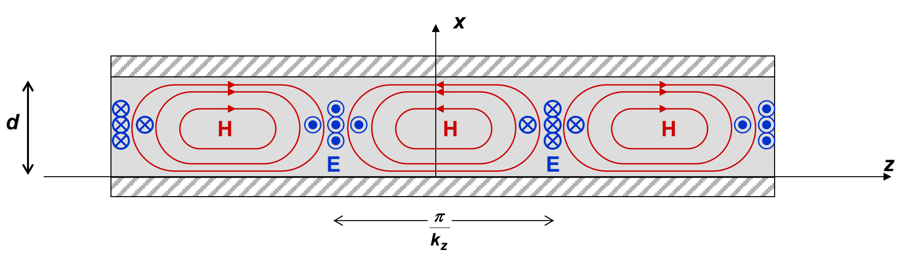
$$\vec{H}(\vec{r}) = \frac{jE_o}{\omega\,\mu_o}\left[\hat{z}\,\frac{m\pi}{d}\cos\left(\frac{m\pi}{d}x\right) + \hat{x}\; j\,k_z \sin\left(\frac{m\pi}{d}x\right)\right] e^{-j k_z z}$$

Note that the perfect metal boundary condition for the magnetic field is automatically satisfied i.e:

$$H_x(\vec{r})\big|_{x=0} = H_x(\vec{r})\big|_{x=d} = 0$$

TE Guided Modes – Field Profiles

The **E-field** and **H-field** lines for the TE_1 mode are shown below:

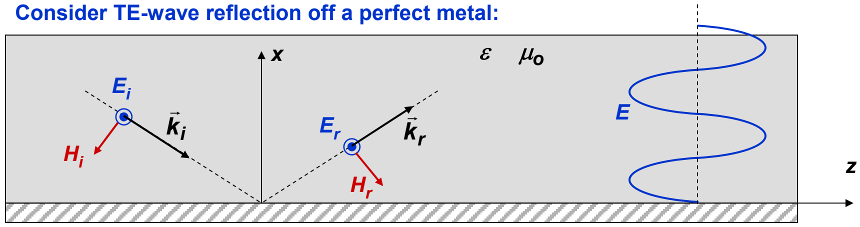


$$\vec{E}(\vec{r}) = \hat{y}\, E_o \sin\left(\frac{m\pi}{d}x\right) e^{-j k_z z} \qquad \{ m = 1, 2, 3, \ldots\ldots$$

$$\vec{H}(\vec{r}) = \frac{jE_o}{\omega\,\mu_o}\left[\hat{z}\,\frac{m\pi}{d}\cos\left(\frac{m\pi}{d}x\right) + \hat{x}\; j\,k_z \sin\left(\frac{m\pi}{d}x\right)\right] e^{-j k_z z}$$

TE Guided Modes – Another Perspective - I

Consider TE-wave reflection off a perfect metal:



$\vec{k}_i = -k_x\hat{x} + k_z\hat{z}$ $\quad \vec{k}_r = k_x\hat{x} + k_z\hat{z}$

$k_z^2 + k_x^2 = \omega^2 \mu_o \varepsilon$

$\vec{E}(\vec{r})\big|_{x>0} = \hat{y}\ E_i\ e^{-j(-k_x x + k_z z)} + \hat{y}\ \Gamma E_i\ e^{-j(k_x x + k_z z)}$ $\longrightarrow \left\{ \Gamma = -1 \right.$

$\vec{E}(\vec{r})\big|_{x>0} = \hat{y}\ E_i \left[e^{-j(-k_x x + k_z z)} - e^{-j(k_x x + k_z z)} \right]$

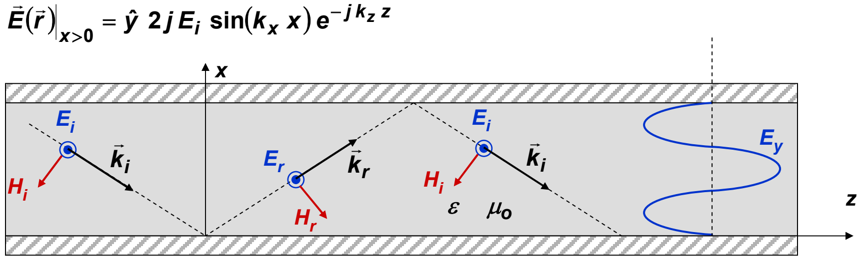
$\Rightarrow \quad \vec{E}(\vec{r})\big|_{x>0} = \hat{y}\ 2j E_i \sin(k_x x)\, e^{-j k_z z}$

Notice the "sine" variation of the y-component of the E-field

TE Guided Modes – Another Perspective - II

If another top metal plate is placed at one of the nodes of the "sine" function then this additional metal plate will not disturb the field

$\vec{E}(\vec{r})\big|_{x>0} = \hat{y}\ 2j E_i \sin(k_x x)\, e^{-j k_z z}$

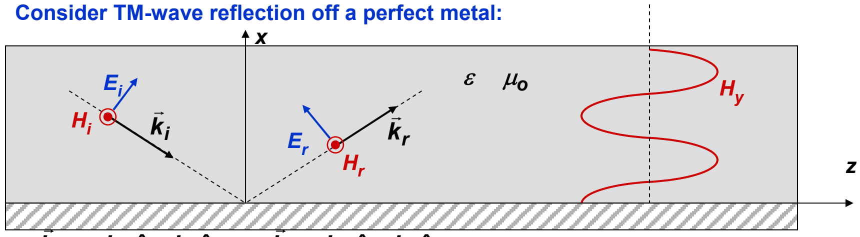


$\vec{k}_i = -k_x\hat{x} + k_z\hat{z}$ $\quad \vec{k}_r = k_x\hat{x} + k_z\hat{z}$

This is exactly what guided TE modes are – TE-waves bouncing back and fourth between two metal plates and propagating in the z-direction !

TM Guided Modes - I

Consider TM-wave reflection off a perfect metal:



$\vec{k}_i = -k_x\hat{x} + k_z\hat{z}$ $\quad$ $\vec{k}_r = k_x\hat{x} + k_z\hat{z}$

$$k_z^2 + k_x^2 = \omega^2\,\mu_o\,\varepsilon$$

$$\vec{H}(\vec{r})\Big|_{x>0} = \hat{y}\;H_i\,e^{-j(-k_x x + k_z z)} + \hat{y}\;\Gamma^{TM} H_i\,e^{-j(k_x x + k_z z)} \longrightarrow \left\{\Gamma^{TM} = +1\right.$$

$$\vec{H}(\vec{r})\Big|_{x>0} = \hat{y}\;H_i\left[e^{-j(-k_x x + k_z z)} + e^{-j(k_x x + k_z z)}\right]$$

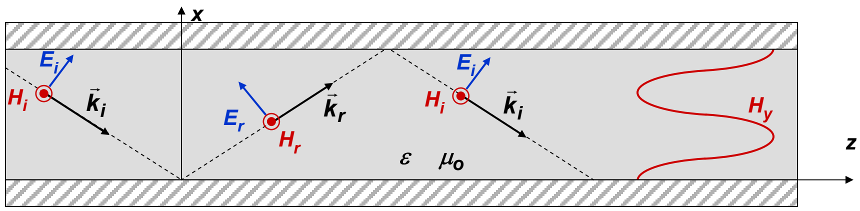
$$\Rightarrow\quad \vec{H}(\vec{r})\Big|_{x>0} = \hat{y}\;2H_i\cos(k_x\,x)\,e^{-j k_z z}$$

Notice the "cosine" variation of the y-component of the H-field

TM Guided Modes - II

If another top metal plate is placed at the maximum points of the "cosine" function then this additional metal plate will not disturb the field

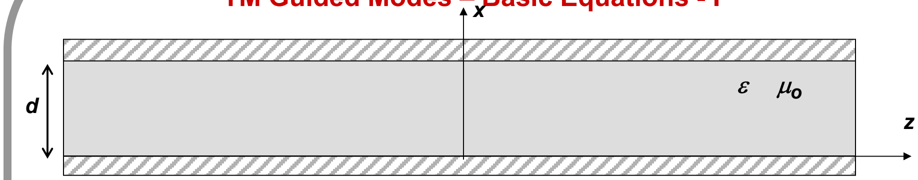
$$\vec{H}(\vec{r})\Big|_{x>0} = \hat{y}\;2H_i\cos(k_x\,x)\,e^{-j k_z z}$$



$\vec{k}_i = -k_x\hat{x} + k_z\hat{z}$ $\quad$ $\vec{k}_r = k_x\hat{x} + k_z\hat{z}$

This is exactly what guided TM modes are – TM-waves bouncing back and fourth between two metal plates and propagating in the z-direction !

TM Guided Modes – Basic Equations - I



Need to solve the equation: $\nabla^2 \vec{H}(\vec{r}) = -\omega^2 \mu_o \varepsilon \vec{H}(\vec{r})$

Assume the solution form: $\vec{H}(\vec{r}) = \hat{y}\, G(x)\, e^{-j k_z z}$

It represents a TM guided wave (Transverse Magnetic) since the direction of H-field is transverse to the direction of wave propagation

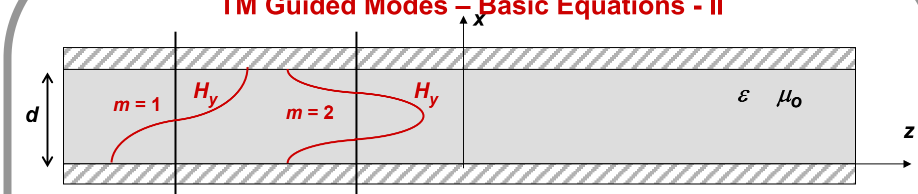
Plugging the assumed solution into the equation gives:

$$\nabla^2 \vec{H}(\vec{r}) = -\omega^2 \mu_o \varepsilon \vec{H}(\vec{r})$$

$$\Rightarrow \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)\vec{H}(\vec{r}) = -\omega^2 \mu_o \varepsilon \vec{H}(\vec{r})$$

$$\Rightarrow -\frac{\partial^2 G(x)}{\partial x^2} = \left(\omega^2 \mu_o \varepsilon - k_z^2\right) G(x)$$

TM Guided Modes – Basic Equations - II



Need to solve: $-\frac{\partial^2 G(x)}{\partial x^2} = \left(\omega^2 \mu_o \varepsilon - k_z^2\right) G(x)$

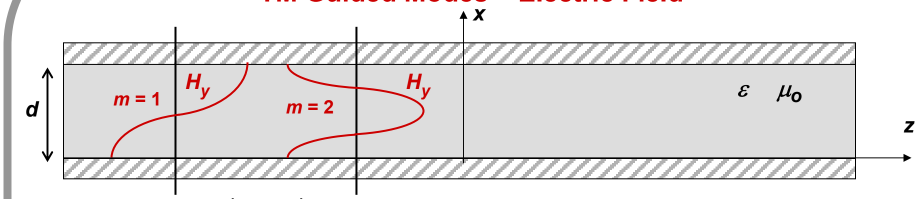
Solution is: $G(x) = H_o \cos(k_x x)$ } Motivation for this is obtained from the TM-wave reflection analysis discussed earlier

$k_x = m\frac{\pi}{d}$ where : $m = 0, 1, 2, 3, \ldots\ldots$

Solution becomes: $G(x) = H_o \cos\left(\frac{m\pi}{d} x\right)$

And: $\boxed{\vec{H}(\vec{r}) = \hat{y}\, H_o \cos\left(\frac{m\pi}{d} x\right) e^{-j k_z z}}$ $\{\, m = 0, 1, 2, 3, \ldots\ldots$

TM Guided Modes – Electric Field



$$\vec{H}(\vec{r}) = \hat{y}\, H_o \cos\left(\frac{m\pi}{d}x\right) e^{-j k_z z} \qquad \{\, m = 0, 1, 2, 3, \ldots\ldots$$

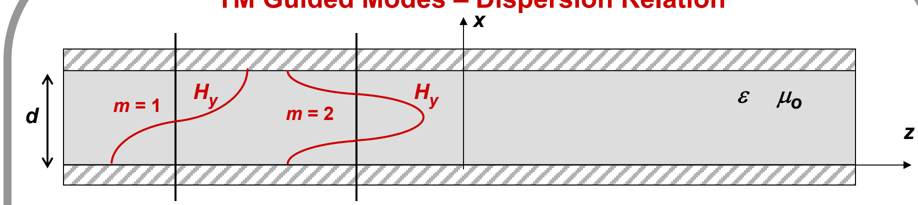
Electric field is given by the equation: $\nabla \times \vec{H}(\vec{r}) = j\,\omega\,\varepsilon\,\vec{E}(\vec{r})$

$$\vec{E}(\vec{r}) = -\frac{jH_o}{\omega\varepsilon}\left[-\hat{z}\,\frac{m\pi}{d}\sin\left(\frac{m\pi}{d}x\right) + \hat{x}\, j\, k_z \cos\left(\frac{m\pi}{d}x\right)\right] e^{-j k_z z}$$

Note that the perfect metal boundary condition for the electric field is automatically satisfied, i.e.:

$$E_z(\vec{r})\big|_{x=0} = E_z(\vec{r})\big|_{x=d} = 0$$

TM Guided Modes – Dispersion Relation



$$\vec{H}(\vec{r}) = \hat{y}\, H_o \cos\left(\frac{m\pi}{d}x\right) e^{-j k_z z} \qquad \{\, m = 0, 1, 2, 3, \ldots\ldots$$

Different "m" values correspond to different TM modes – labeled as TM_m modes

The equation: $-\dfrac{\partial^2 G(x)}{\partial x^2} = \left(\omega^2 \mu_o \varepsilon - k_z^2\right) G(x)$ implies:

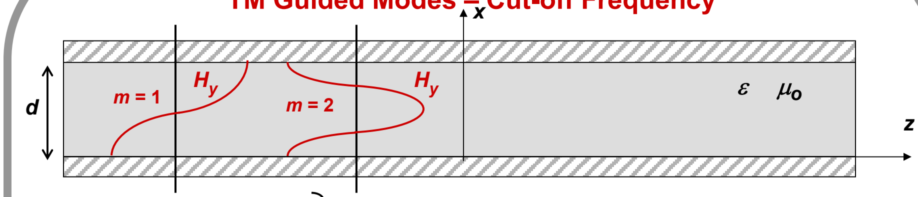
$$k_z^2 + k_x^2 = \omega^2 \mu_o \varepsilon$$

$$\Rightarrow \quad k_z^2 + \left(\frac{m\pi}{d}\right)^2 = \omega^2 \mu_o \varepsilon$$

$$\Rightarrow \quad k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{d}\right)^2}$$

Dispersion relation for TM_m waveguide mode

TM Guided Modes – Cut-off Frequency



$$k_z = \sqrt{\omega^2 \mu_o \varepsilon - \left(\frac{m\pi}{d}\right)^2}$$

Dispersion relation for TM_m guided mode

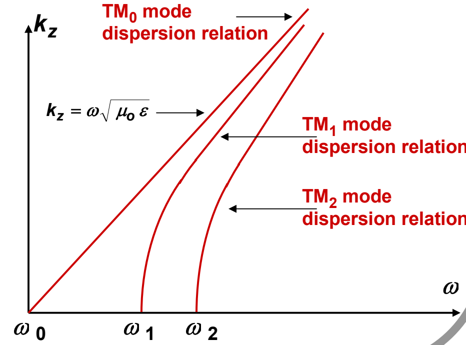
For the TM_m mode, if the frequency ω is less than:

$$\frac{1}{\sqrt{\mu_o \varepsilon}}\left(\frac{m\pi}{d}\right)$$

Then k_z becomes entirely imaginary and the mode does not propagate (but decays exponentially with distance)

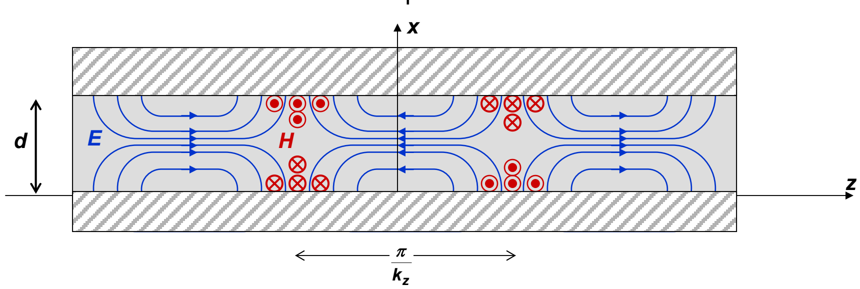
⇒ Cut-off frequency for TM_m mode:

$$\omega_m = \frac{1}{\sqrt{\mu_o \varepsilon}}\left(\frac{m\pi}{d}\right)$$



TM Guided Modes – Field Profiles

The E-field and H-field lines for the TM_1 mode are shown below:



$$\vec{H}(\vec{r}) = \hat{y}\, H_o \cos\left(\frac{m\pi}{d}x\right) e^{-j k_z z} \qquad \{\, m = 0, 1, 2, 3, \ldots\ldots$$

$$\vec{E}(\vec{r}) = -\frac{jH_o}{\omega\varepsilon}\left[-\hat{z}\,\frac{m\pi}{d}\sin\left(\frac{m\pi}{d}x\right) + \hat{x}\, j k_z \cos\left(\frac{m\pi}{d}x\right)\right] e^{-j k_z z}$$

TM_0 Guided Mode – Field Profiles

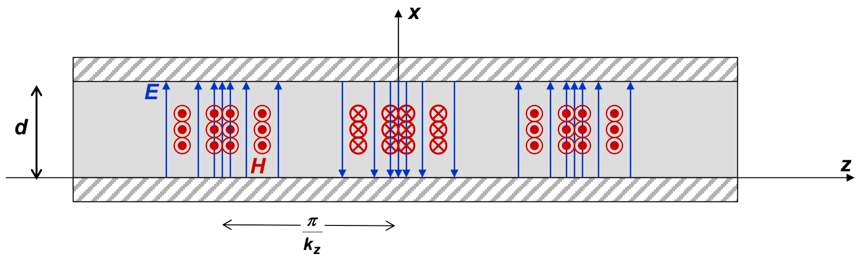
The E-field and H-field for the TM_0 mode are:

$$\vec{H}(\vec{r}) = \hat{y}\, H_o\, e^{-j k_z z}$$

$$\vec{E}(\vec{r}) = \hat{x}\, \frac{k_z}{\omega \varepsilon} H_o\, e^{-j k_z z}$$

Note that fields are not a function of "x"

The E-field and H-field lines for the TM_0 mode are shown below:



The TM_0 mode is just the TEM mode that we worked with when dealing with transmission lines !