Lecture 18

Reflection and Transmission of Waves at Interfaces

In this lecture you will learn:

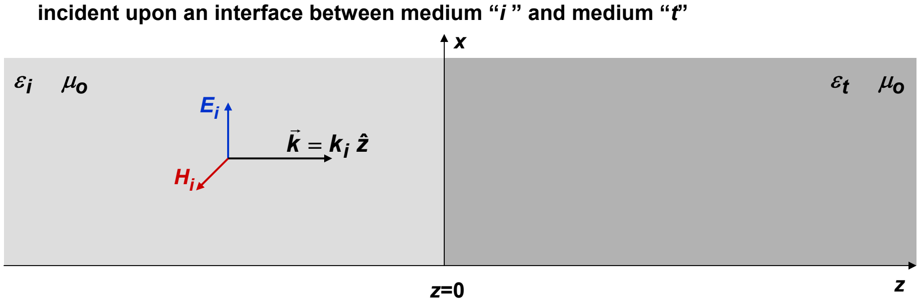
- What happens when waves strike an interface between two different media
- Reflection and transmission of waves at interfaces
- Application of E-field and H-field boundary conditions

Waves at Interfaces

Consider a plane wave given by:

$$\vec{E}(\vec{r}) = \hat{x}\, E_i\, e^{-j k_i z}$$

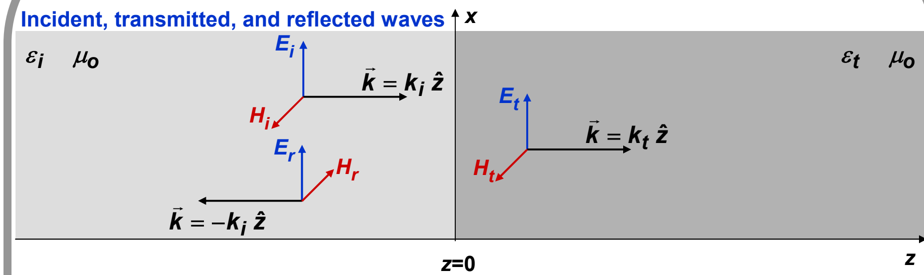
incident upon an interface between medium "*i*" and medium "*t*"



- When the incident wave strikes the interface, it generates a transmitted wave and a reflected wave
- We need to find the amplitudes of these reflected and transmitted waves

Waves at Interfaces – E-Fields

Incident, transmitted, and reflected waves



For $z < 0$ the total E-field is:

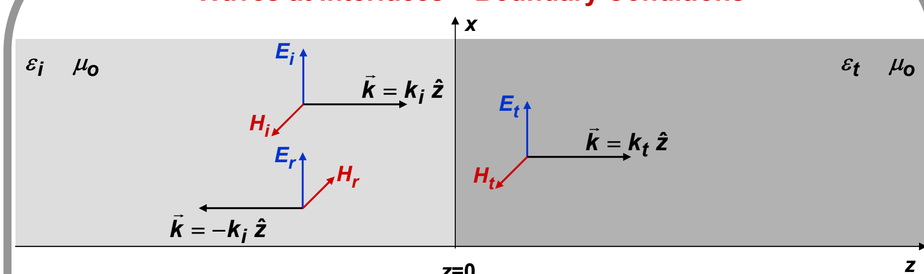
$$\vec{E}(\vec{r})\Big|_{z<0} = \hat{x}\,E_i\,e^{-j\,k_i\,z} + \hat{x}\,\underbrace{E_r}_{?}\,e^{+j\,k_i\,z} \longrightarrow \left\{ k_i = \omega\sqrt{\mu_o\,\varepsilon_i} \right.$$

For $z > 0$ the total E-field is:

$$\vec{E}(\vec{r})\Big|_{z>0} = \hat{x}\,\underbrace{E_t}_{?}\,e^{-j\,k_t\,z} \longrightarrow \left\{ k_t = \omega\sqrt{\mu_o\,\varepsilon_t} \right.$$

In the above equations, E_r and E_t are the unknowns that we need to find in terms of the incident field amplitude E_i

Waves at Interfaces – Boundary Conditions



For $z < 0$: $\vec{E}(\vec{r})\Big|_{z<0} = \hat{x}\,E_i\,e^{-j\,k_i\,z} + \hat{x}\,E_r\,e^{+j\,k_i\,z}$

For $z > 0$: $\vec{E}(\vec{r})\Big|_{z>0} = \hat{x}\,E_t\,e^{-j\,k_t\,z}$

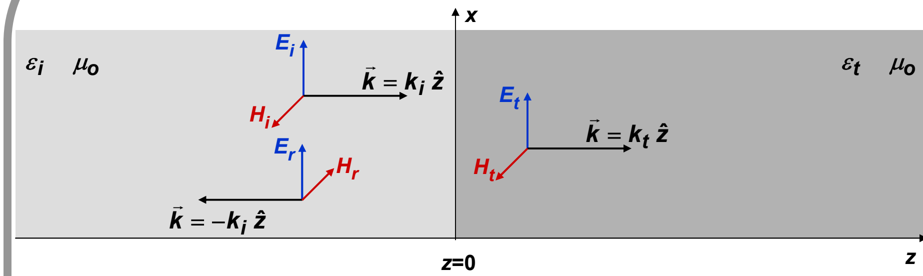
Use boundary conditions:

(1) At $z = 0$ the E-field parallel to the interface must be continuous

This gives:

$$\boxed{E_i + E_r = E_t}$$

Waves at Interfaces – H-fields



(2) At $z = 0$ the H-field parallel to the interface must be continuous (no surface currents)

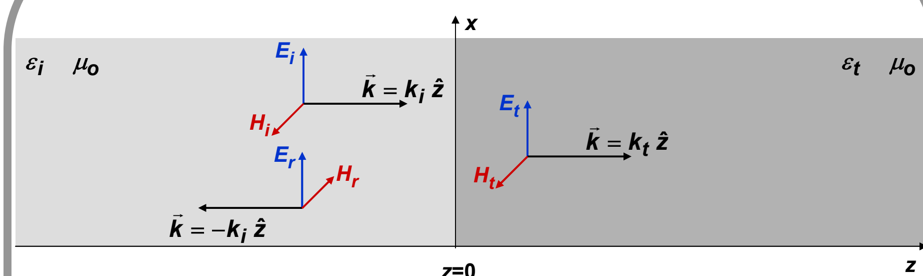
For $z < 0$: $$\vec{H}(\vec{r})\Big|_{z<0} = \hat{y}\,\frac{E_i}{\eta_i}\,e^{-j k_i z} - \hat{y}\,\frac{E_r}{\eta_i}\,e^{+j k_i z}$$

For $z > 0$: $$\vec{H}(\vec{r})\Big|_{z>0} = \hat{y}\,\frac{E_t}{\eta_t}\,e^{-j k_t z}$$

This gives:
$$\frac{E_i}{\eta_i} - \frac{E_r}{\eta_i} = \frac{E_t}{\eta_t}$$

The other equation was:
$$E_i + E_r = E_t$$

Waves at Interfaces – Reflection and Transmission Coefficients



The solution is:

Transmission coefficient
$$T = \frac{E_t}{E_i} = \frac{2\frac{\eta_t}{\eta_i}}{\frac{\eta_t}{\eta_i}+1} = \frac{2\frac{k_i}{k_t}}{\frac{k_i}{k_t}+1} = \frac{2\frac{n_i}{n_t}}{\frac{n_i}{n_t}+1}$$

Reflection coefficient
$$\Gamma = \frac{E_r}{E_i} = \frac{\frac{\eta_t}{\eta_i}-1}{\frac{\eta_t}{\eta_i}+1} = \frac{\frac{k_i}{k_t}-1}{\frac{k_i}{k_t}+1} = \frac{\frac{n_i}{n_t}-1}{\frac{n_i}{n_t}+1}$$

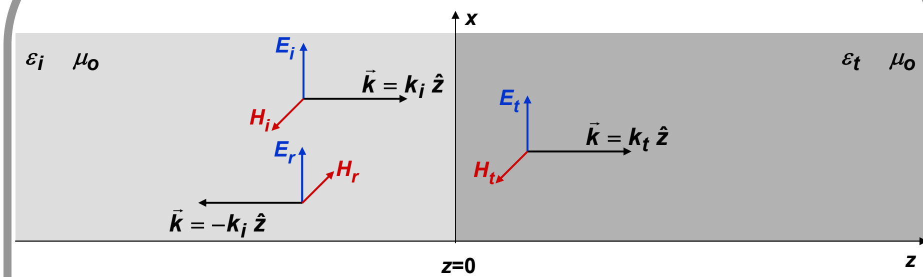
These are three different ways of writing the same results:

i) Using the impedances

ii) Using the wavevectors

iii) Using the refractive indices

Waves at Interfaces – Power Flow



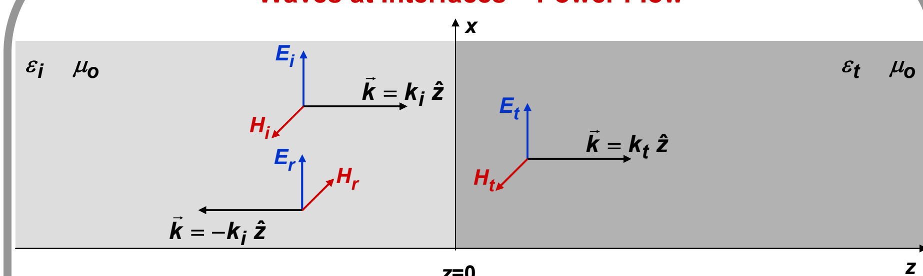
Power at the interface for $z < 0$:

$$\vec{E}(\vec{r})\Big|_{z<0} = \hat{x}\,E_i\,e^{-j k_i z} + \hat{x}\,\Gamma E_i\,e^{+j k_i z} \qquad \vec{H}(\vec{r})\Big|_{z<0} = \hat{y}\,\frac{E_i}{\eta_i}\,e^{-j k_i z} - \hat{y}\,\Gamma\frac{E_i}{\eta_i}\,e^{+j k_i z}$$

$$\left\langle \vec{S}(\vec{r},t)\right\rangle\Big|_{z=0} = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right]_{z=0}$$
$$= \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right]_{z=0}$$
$$= \hat{z}\,\frac{|E_i|^2}{2}\,\mathrm{Re}\left(\frac{1}{\eta_i^*}\right)\left[1-|\Gamma|^2\right]$$

- This is the power per unit area just to the left of the interface
- It has separate contributions from the incident and the reflected waves

Waves at Interfaces – Power Flow



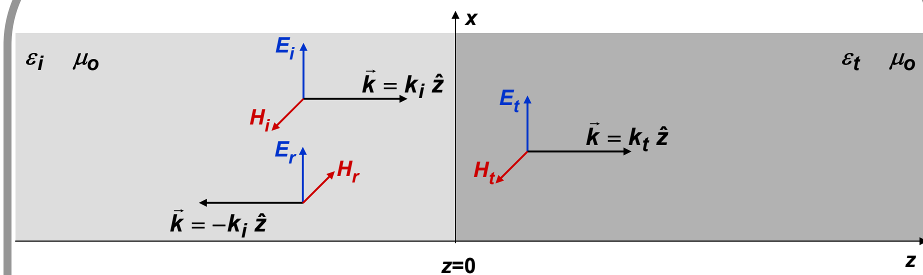
Power at the interface for $z > 0$:

$$\vec{E}(\vec{r})\Big|_{z>0} = \hat{x}\,E_t\,e^{-j k_t z} \qquad \vec{H}(\vec{r})\Big|_{z>0} = \hat{y}\,\frac{E_t}{\eta_t}\,e^{-j k_t z}$$

$$\left\langle \vec{S}(\vec{r},t)\right\rangle\Big|_{z=0} = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right]_{z=0}$$
$$= \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right]_{z=0}$$
$$= \hat{z}\,\frac{|E_i|^2}{2}\,\mathrm{Re}\left(\frac{1}{\eta_t^*}\right)|T|^2$$

- This is the power per unit area just to the right of the interface

Waves at Interfaces – Conservation of Power



Conservation of power at the interface requires:

Net power in the +*z*-direction just to the left of the interface = Net power in the +*z*-direction just to the right of the interface

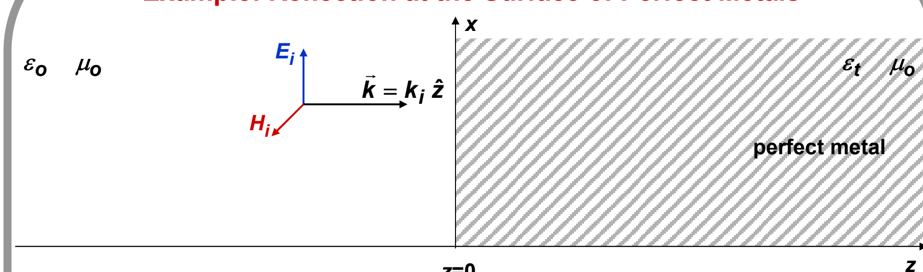
$$\Rightarrow \quad \frac{|E_i|^2}{2}\,\text{Re}\left(\frac{1}{\eta_i^*}\right)\left[1-|\Gamma|^2\right] = \frac{|E_i|^2}{2}\,\text{Re}\left(\frac{1}{\eta_t^*}\right)|T|^2$$

or:

$$\text{Re}\left(\frac{1}{\eta_i^*}\right)\left[1-|\Gamma|^2\right] = \text{Re}\left(\frac{1}{\eta_t^*}\right)|T|^2$$

The last equation is indeed true if the earlier found values of Γ and T are used

Example: Reflection at the Surface of Perfect Metals



For perfect metals:

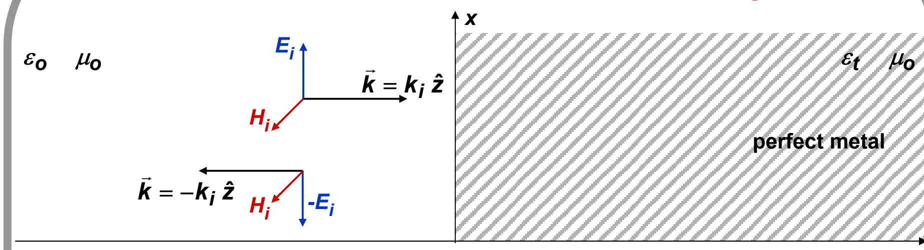
$$\varepsilon_t(\omega) = \varepsilon_o\left(1 - j\frac{\sigma}{\omega\,\varepsilon_o}\right)\Bigg|_{\sigma\to\infty} \quad \Rightarrow \quad |\varepsilon_t(\omega)| \to \infty$$

$$\eta_t(\omega) = \sqrt{\frac{\mu_o}{\varepsilon_t(\omega)}} \quad \Rightarrow \quad |\eta_t(\omega)| \to 0$$

Reflection and transmission coefficients for perfect metal and free-space interface:

$$T = \frac{E_t}{E_i} = \frac{2\dfrac{\eta_t}{\eta_o}}{\dfrac{\eta_t}{\eta_o}+1} \to 0 \qquad \Gamma = \frac{E_r}{E_i} = \frac{\dfrac{\eta_t}{\eta_o}-1}{\dfrac{\eta_t}{\eta_o}+1} \to -1$$

Reflection at the Surface of Perfect Metals – Standing Waves



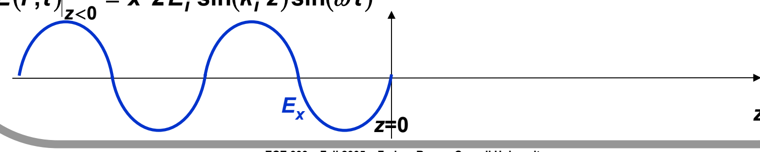
$$\vec{E}(\vec{r})\big|_{z<0} = \hat{x}\,E_i\,e^{-j k_i z} + \hat{x}\,\Gamma E_i\,e^{+j k_i z} = -\hat{x}\,2j E_i \sin(k_i z)$$

$$\vec{H}(\vec{r})\big|_{z<0} = \hat{y}\,\frac{E_i}{\eta_o}\,e^{-j k_i z} - \hat{y}\,\Gamma\frac{E_i}{\eta_o}\,e^{+j k_i z} = \hat{y}\,\frac{2E_i}{\eta_o}\cos(k_i z)$$

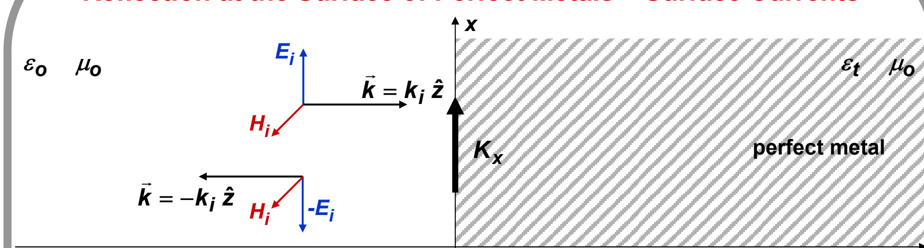
→ Standing waves

The interference between incident and reflected waves gives rise to standing waves

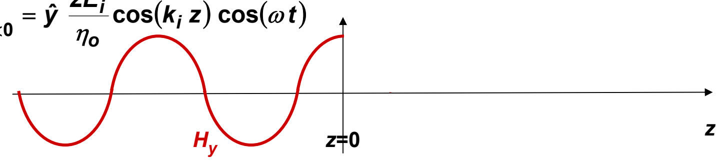
$$\vec{E}(\vec{r},t)\big|_{z<0} = \hat{x}\,2E_i\sin(k_i z)\sin(\omega t)$$



Reflection at the Surface of Perfect Metals – Surface Currents



$$\vec{H}(\vec{r},t)\big|_{z<0} = \hat{y}\,\frac{2E_i}{\eta_o}\cos(k_i z)\cos(\omega t)$$

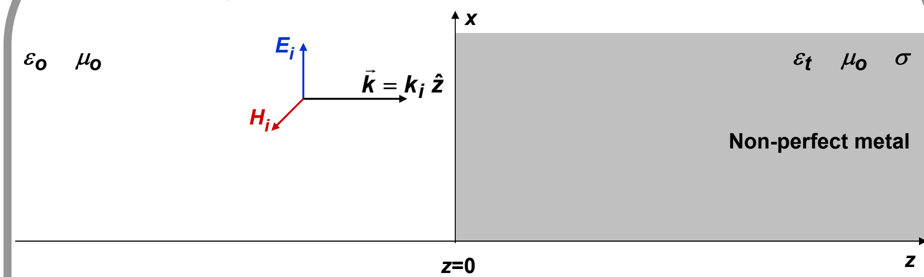


- The magnetic field is parallel to the surface and peaks at the metal surface
- There must be surface currents (recall the boundary conditions for the magnetic field)

$$H_y(z=0,t)\big|_{outside} - \underbrace{H_y(z=0,t)\big|_{inside}}_{0} = K_x(z=0,t)$$

$$\Rightarrow\ \frac{2E_i}{\eta_o}\cos(\omega t) = K_x(z=0,t)$$

Example: Non-Perfect Metals or Good Conductors



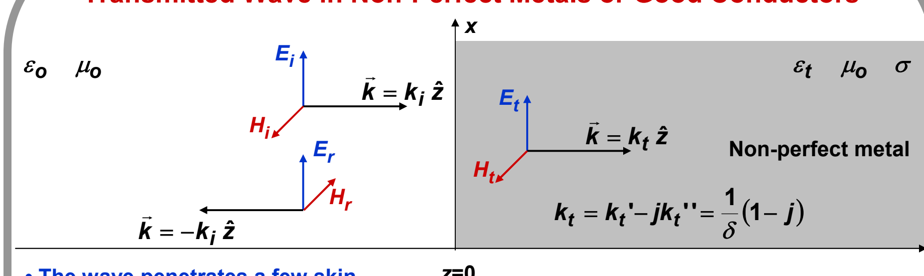
For non-perfect metals:

$$\varepsilon_t(\omega) = \varepsilon\left(1 - j\frac{\sigma}{\omega\,\varepsilon}\right) \qquad \eta_t(\omega) = \sqrt{\frac{\mu_o}{\varepsilon_t(\omega)}}$$

Reflection and transmission coefficients for non-perfect metal are complex numbers:

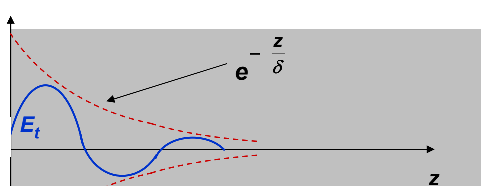
$$T = \frac{E_t}{E_i} = \frac{2\,\dfrac{\eta_t(\omega)}{\eta_o}}{\dfrac{\eta_t(\omega)}{\eta_o} + 1} \qquad \Gamma = \frac{E_r}{E_i} = \frac{\dfrac{\eta_t(\omega)}{\eta_o} - 1}{\dfrac{\eta_t(\omega)}{\eta_o} + 1}$$

Transmitted Wave in Non-Perfect Metals or Good Conductors



- **The wave penetrates a few skin-depths into the non-perfect metal**

$$\vec{E}(\vec{r})\Big|_{z>0} = \hat{x}\,TE_i\,e^{-j\,k_t\,z} = \hat{x}\,TE_i\,e^{-j\frac{z}{\delta}}\,e^{-\frac{z}{\delta}}$$

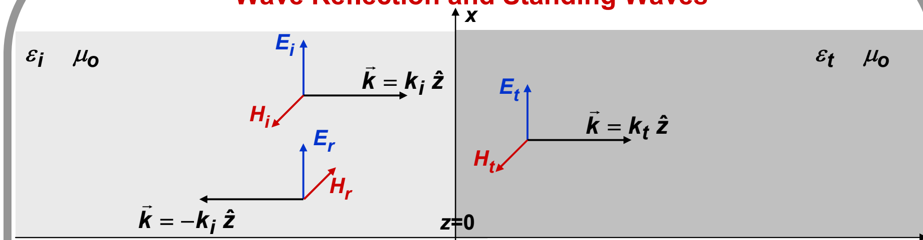


$$k_t'' = \frac{1}{\delta} = \sqrt{\frac{\omega\,\mu_o\,\sigma}{2}}$$

- **Current flows within a layer a few skin-depths thick inside the non-perfect metal**

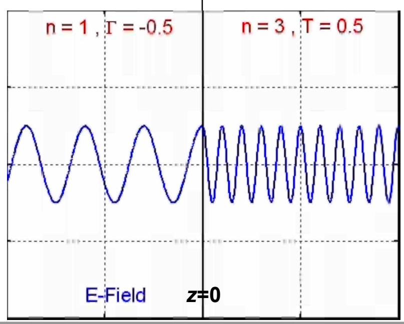
$$\vec{J}(\vec{r}) = \sigma\,\vec{E}(\vec{r}) = \hat{x}\,\sigma\,T\,E_i\,e^{-j\frac{z}{\delta}}\,e^{-\frac{z}{\delta}}$$

Wave Reflection and Standing Waves

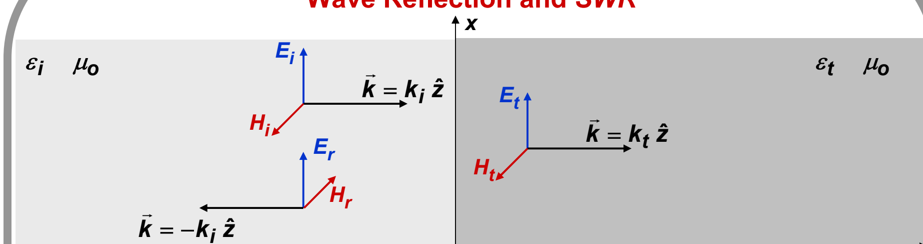


$$\vec{E}(\vec{r})\big|_{z<0} = \hat{x}\,E_i\,e^{-j\,k_i\,z} + \hat{x}\,\Gamma\,E_i\,e^{+j\,k_i\,z}$$

The interference between incident and reflected waves gives rise to standing waves for z<0



Wave Reflection and *SWR*



z=0

$$\vec{E}(\vec{r})\big|_{z<0} = \hat{x}\,E_i\,e^{-j\,k_i\,z} + \hat{x}\,\Gamma\,E_i\,e^{+j\,k_i\,z} \longrightarrow \left\{\text{Let}: \quad \Gamma = |\Gamma|\,e^{j\phi}\right.$$

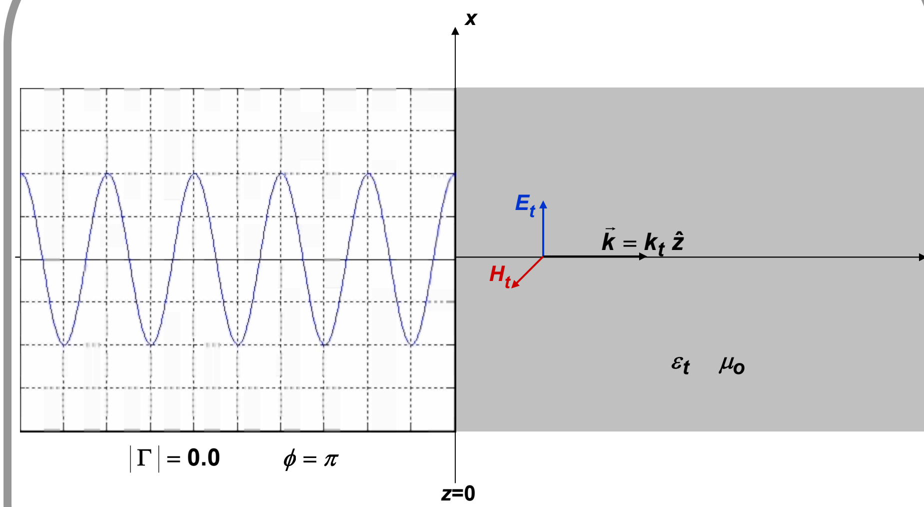
$$\Rightarrow \quad \left|\vec{E}(\vec{r})\big|_{z<0}\right| = |E_i|\,\sqrt{1 + |\Gamma|^2 + 2|\Gamma|\cos(2k_i\,z + \phi)}$$

$$\Rightarrow \quad \left|\vec{E}(z_{max})\right| = |E_i|\,(1 + |\Gamma|) \qquad \left|\vec{E}(z_{min})\right| = |E_i|\,(1 - |\Gamma|)$$

Standing Wave Ratio (SWR): $SWR = \dfrac{\left|\vec{E}(z_{max})\right|}{\left|\vec{E}(z_{min})\right|} = \dfrac{1 + |\Gamma|}{1 - |\Gamma|}$

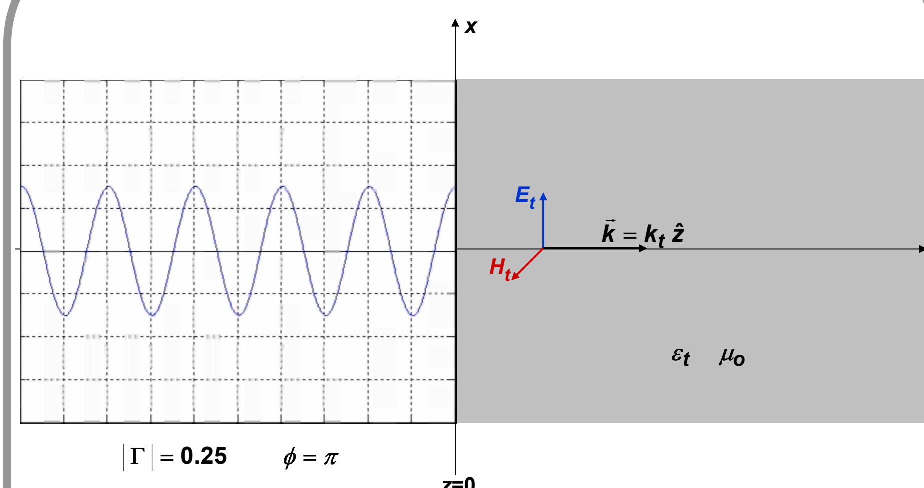
***SWR* is found by taking the ratio of the magnitude of the maximum field value (wherever that occurs) to the minimum field value (wherever that occurs)**

Wave Reflection and *SWR* - No Reflection



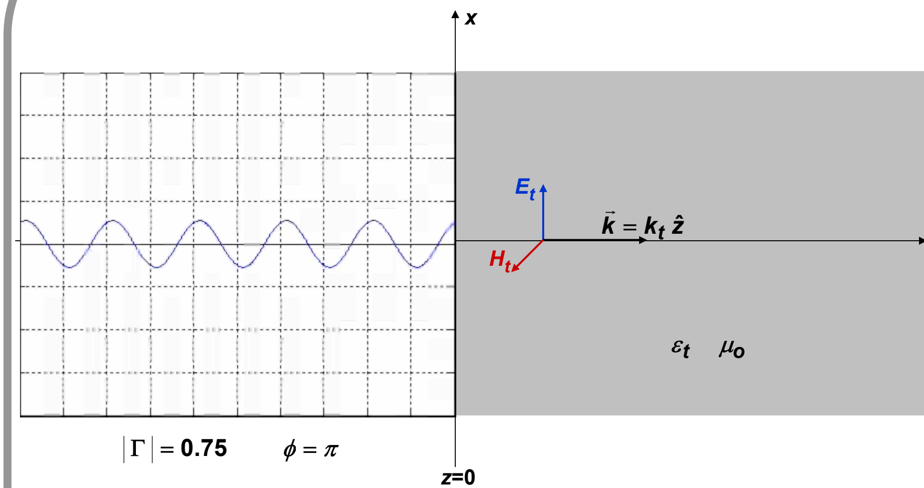
$$SWR = \frac{|\vec{E}(z_{max})|}{|\vec{E}(z_{min})|} = \frac{1+|\Gamma|}{1-|\Gamma|} = 1$$

Wave Reflection and *SWR* – Small Reflection



$$SWR = \frac{|\vec{E}(z_{max})|}{|\vec{E}(z_{min})|} = \frac{1+|\Gamma|}{1-|\Gamma|} = 1.66$$

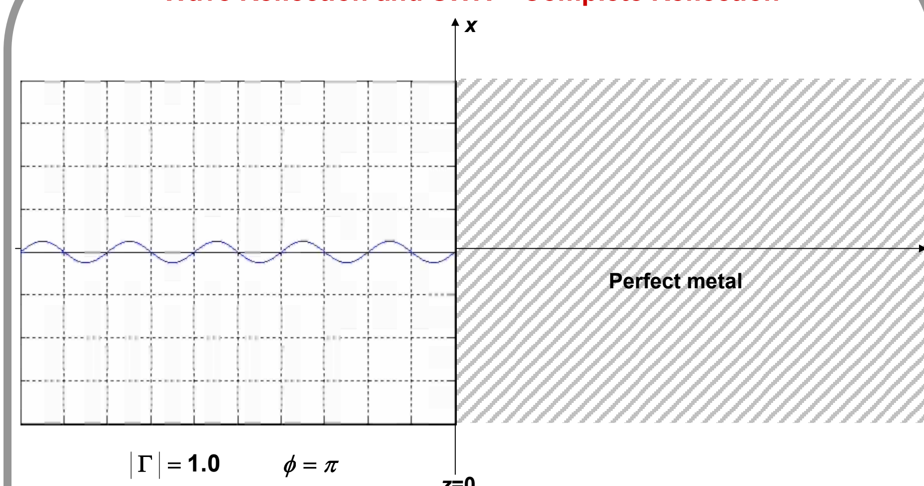
Wave Reflection and *SWR* – Large Reflection



$|\Gamma| = 0.75 \qquad \phi = \pi$

$$SWR = \frac{|\vec{E}(z_{max})|}{|\vec{E}(z_{min})|} = \frac{1+|\Gamma|}{1-|\Gamma|} = 7$$

Wave Reflection and *SWR* – Complete Reflection



$|\Gamma| = 1.0 \qquad \phi = \pi$

$$SWR = \frac{|\vec{E}(z_{max})|}{|\vec{E}(z_{min})|} = \frac{1+|\Gamma|}{1-|\Gamma|} = \infty$$