Lecture 16

Waves in Isotropic Media: Dielectrics and Conductors

In this lecture you will learn:

- **Wave propagation in dielectric media**
- **Waves propagation in conductive media**

Review: Plane Waves in Free Space

Faraday's Law:

$$\nabla\times\vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})$$

Ampere's Law:

$$\nabla\times\vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\,\omega\,\varepsilon_o\,\vec{E}(\vec{r})$$

Complex Wave Equation: **Assume:** $\vec{J}(\vec{r}) = \rho(\vec{r}) = 0$

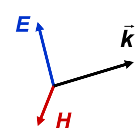
$$\nabla\times\nabla\times\vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\nabla\times\vec{H}(\vec{r}) = \omega^2\,\mu_o\,\varepsilon_o\,\vec{E}(\vec{r})$$

$$\Rightarrow\quad \nabla\big(\overset{0}{\nabla.\vec{E}(\vec{r})}\big) - \nabla^2\,\vec{E}(\vec{r}) = \omega^2\,\mu_o\,\varepsilon_o\,\vec{E}(\vec{r})$$

$$\boxed{\Rightarrow\quad \nabla^2\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon_o\,\vec{E}(\vec{r})}$$

For a plane wave in free space we know the E-field and H-field phasors to be:

$$\left.\begin{aligned}\vec{E}(\vec{r}) &= \hat{n}\,E_o\,e^{-j\,\vec{k}.\vec{r}}\\ \vec{H}(\vec{r}) &= \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\,e^{-j\,\vec{k}.\vec{r}}\end{aligned}\right\}\qquad \begin{aligned}k &= \omega\sqrt{\mu_o\,\varepsilon_o} = \frac{\omega}{c}\\ \eta_o &= \sqrt{\frac{\mu_o}{\varepsilon_o}} \approx 377\,\Omega\end{aligned}$$

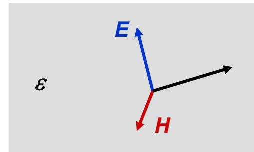


Waves in a Dielectric Medium – Wave Equation

Suppose we have a plane wave of the form,

$$\vec{E}(\vec{r}) = \hat{n}\, E_o\, e^{-j\,\vec{k}.\vec{r}}$$

traveling in an infinite dielectric medium with permittivity ε



What is different from wave propagation in free space?

Faraday's Law:

$$\nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})$$

Ampere's Law:

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\,\omega\,\varepsilon\,\vec{E}(\vec{r})$$

Complex Wave Equation: **Assume:** $\vec{J}(\vec{r}) = \rho(\vec{r}) = 0$

$$\nabla \times \nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\nabla \times \vec{H}(\vec{r}) = \omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

$$\Rightarrow \quad \nabla\left(\underbrace{\nabla.\vec{E}(\vec{r})}_{0}\right) - \nabla^2\,\vec{E}(\vec{r}) = \omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

$$\boxed{\Rightarrow \quad \nabla^2\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})} \longrightarrow$$

compare with the complex wave equation in free space

$$\nabla^2\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon_o\,\vec{E}(\vec{r})$$

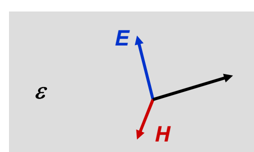
Waves in a Dielectric Medium – Dispersion Relation

Substitute the plane wave solution:

$$\vec{E}(\vec{r}) = \hat{n}\, E_o\, e^{-j\,\vec{k}.\vec{r}}$$

in the complex wave equation:

$$\nabla^2\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$



To get:

$$\nabla^2\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

$$\Rightarrow \quad -\vec{k}.\vec{k}\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon\,\vec{E}(\vec{r})$$

$$\Rightarrow \quad k^2 = \omega^2\,\mu_o\,\varepsilon$$

$$\boxed{\Rightarrow \quad k = \omega\sqrt{\mu_o\,\varepsilon}} \longrightarrow$$

compare with $k = \omega\sqrt{\mu_o\,\varepsilon_o}$ for waves in free space

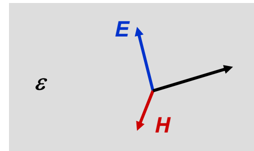
Refractive Index:

Define refractive index "*n*" of a dielectric medium as: $n = \sqrt{\dfrac{\varepsilon}{\varepsilon_o}}$

Waves in a Dielectric Medium – Velocity

Plane wave:

$$\vec{E}(\vec{r}) = \hat{n}\, E_o\, e^{-j\vec{k}.\vec{r}}$$



Dispersion relation:

$$k = \omega\sqrt{\mu_o\,\varepsilon}$$

$$\Rightarrow \quad k = \omega\sqrt{\mu_o\,\varepsilon_o}\sqrt{\frac{\varepsilon}{\varepsilon_o}} \longrightarrow \left\{ n = \sqrt{\frac{\varepsilon}{\varepsilon_o}} \right.$$

$$\boxed{\Rightarrow \quad k = \omega\frac{n}{c}} \longrightarrow \left\{ \text{compare with } k = \frac{\omega}{c} \text{ for waves in free space} \right.$$

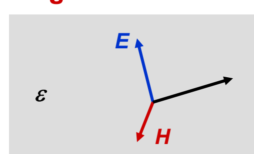
The velocity of waves in a dielectric medium is reduced from the velocity of waves in free space by the refractive index

- **Velocity of waves in free space: c**
- **Velocity of waves in dielectric medium of refractive index n: $\dfrac{c}{n}$**

Waves in a Dielectric Medium - Wavelength

Plane wave in a dielectric medium:

$$\vec{E}(\vec{r}) = \hat{n}\, E_o\, e^{-j\vec{k}.\vec{r}}$$



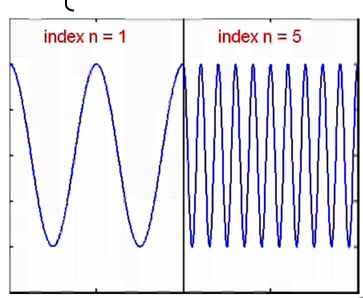
Dispersion relation: $k = \omega\dfrac{n}{c}$

But the magnitude of the wavevector is related to the wavelength by the relation: $k = \dfrac{2\pi}{\lambda}$

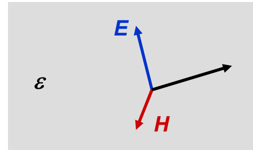
So for a dielectric medium we get:

$$\lambda = \frac{2\pi c}{\omega n} \longrightarrow \left\{ \text{compare with } \lambda = \frac{2\pi c}{\omega} \text{ for waves in free space} \right.$$

The wavelength of plane waves in a dielectric medium is reduced from the wavelength of plane waves of the same frequency in free space by the refractive index



Waves in a Dielectric Medium – Magnetic Field



Plane wave:

$$\vec{E}(\vec{r}) = \hat{n}\, E_o\, e^{-j\,\vec{k}.\vec{r}}$$

Calculate the magnetic field:

$$\vec{H}(\vec{r}) = \frac{j}{\omega\,\mu_o} \nabla \times \vec{E}(\vec{r})$$

$$\Rightarrow \quad \vec{H}(\vec{r}) = \left(\hat{k} \times \hat{n}\right) \frac{k}{\omega\,\mu_o} E_o\, e^{-j\,\vec{k}.\vec{r}}$$

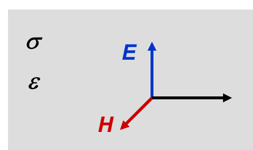
$$\Rightarrow \quad \vec{H}(\vec{r}) = \left(\hat{k} \times \hat{n}\right) \frac{k}{\omega\,\mu_o} E_o\, e^{-j\,\vec{k}.\vec{r}}$$

$$\Rightarrow \quad \vec{H}(\vec{r}) = \left(\hat{k} \times \hat{n}\right) \frac{E_o}{\eta}\, e^{-j\,\vec{k}.\vec{r}}$$

The wave impedance also changes

$$\eta = \sqrt{\frac{\mu_o}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o}\frac{\varepsilon_o}{\varepsilon}} = \frac{\eta_o}{n}$$

Waves in a Conductive Medium – Complex Permittivity



Suppose we have a plane wave of the form,

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j\,k\,z}$$

traveling in an infinite medium with conductivity σ and permittivity ε

Faraday's Law:

$$\nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})$$

Ampere's Law: **Now** $\vec{J}(\vec{r}) \neq 0$

$$\begin{aligned} \nabla \times \vec{H}(\vec{r}) &= \vec{J}(\vec{r}) + j\,\omega\,\varepsilon\,\vec{E}(\vec{r}) \\ &= \sigma\,\vec{E}(\vec{r}) + j\,\omega\,\varepsilon\,\vec{E}(\vec{r}) \\ &= j\,\omega\,\varepsilon_{eff}(\omega)\,\vec{E}(\vec{r}) \end{aligned}$$

$$\varepsilon_{eff}(\omega) = \varepsilon\left(1 - j\frac{\sigma}{\omega\,\varepsilon}\right)$$

Complex Wave Equation:

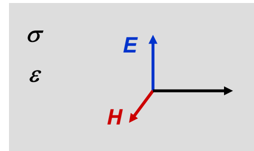
$$\Rightarrow \quad \nabla^2\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon_{eff}(\omega)\,\vec{E}(\vec{r})$$

The effect of conductivity has been absorbed in a complex frequency dependent effective permittivity

Waves in a Conductive Medium – Complex Refractive Index

Plane wave:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$



Dispersion relation:

$$k = \omega \sqrt{\mu_o\, \varepsilon_{eff}(\omega)}$$

$$\Rightarrow \quad k = \omega \sqrt{\mu_o\, \varepsilon_o} \sqrt{\frac{\varepsilon_{eff}(\omega)}{\varepsilon_o}} \longrightarrow \left\{ n_{eff}(\omega) = \sqrt{\frac{\varepsilon_{eff}(\omega)}{\varepsilon_o}} \right.$$

complex refractive index

$$\boxed{\Rightarrow \quad k = \omega \frac{n_{eff}(\omega)}{c}} \longrightarrow$$

compare with $k = \frac{\omega}{c}$ for waves in free space

Since the refractive index is complex: the wavevector k is also complex

$$n_{eff}(\omega) = \sqrt{\frac{\varepsilon}{\varepsilon_o}} \sqrt{1 - j\frac{\sigma}{\omega\,\varepsilon}}$$

$$\boxed{\frac{\sigma}{\omega\,\varepsilon} = \text{loss tangent}}$$

Let: $k = k' - j\,k''$

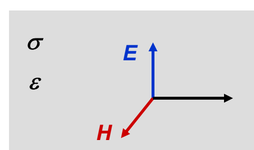
Real part

Imaginary part

Waves in a Conductive Medium – Complex Wavevector

Plane wave:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$



Complex wavevector: $k = k' - j\,k''$

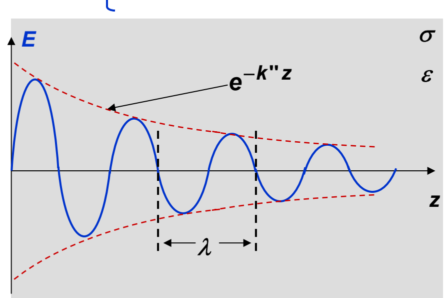
What are the implications of a complex wavevector?

- **Wave decays exponentially with distance as it propagates**

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k' z}\, e^{-k'' z}$$

{ exponential decay



- **The wavelength is related to the real part of the wavevector:**

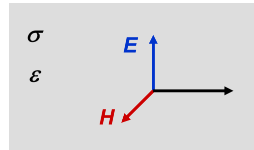
$$k' = \frac{2\pi}{\lambda}$$

Waves in a Conductive Medium – Magnetic Field

Plane wave:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$

Calculate the magnetic field:



$$\vec{H}(\vec{r}) = \frac{j}{\omega \mu_o} \nabla \times \vec{E}(\vec{r})$$

$$\Rightarrow \quad \vec{H}(\vec{r}) = \hat{y} \frac{k}{\omega \mu_o} E_o\, e^{-j k z}$$

Don't forget that the wavevector "k" is complex now

$$\Rightarrow \quad \vec{H}(\vec{r}) = \hat{y} \frac{k}{\omega \mu_o} E_o\, e^{-j k z}$$

$$\Rightarrow \quad \vec{H}(\vec{r}) = \hat{y} \frac{E_o}{\eta_{eff}(\omega)} e^{-j k z}$$

The impedance is now also complex

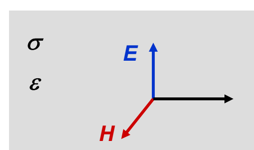
$$\eta_{eff}(\omega) = \sqrt{\frac{\mu_o}{\varepsilon_{eff}(\omega)}}$$

Note: The E-field and the H-field are no longer in phase since $\eta_{eff}(\omega)$ is complex

Waves in a Conductive Medium – Power Flow

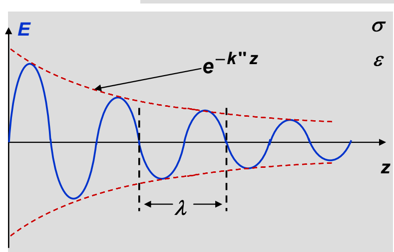
Plane wave:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z} \qquad \vec{H}(\vec{r}) = \hat{y} \frac{E_o}{\eta_{eff}(\omega)} e^{-j k z}$$



$$k = k' - j k''$$

Note that: $\dfrac{1}{\eta_{eff}(\omega)} = \dfrac{k}{\omega \mu_o} = \dfrac{k' - j k''}{\omega \mu_o}$



Poynting vector and time average power per unit area:

$$\langle \vec{S}(\vec{r},t) \rangle = \frac{1}{2} \mathrm{Re}\left[\vec{S}(\vec{r})\right]$$

$$= \frac{1}{2} \mathrm{Re}\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right]$$

$$= \frac{1}{2} \mathrm{Re}\left[\hat{z} \frac{E_o^2}{(\eta_{eff}(\omega))^*} e^{-2k''z}\right] = \hat{z} \frac{E_o^2}{2} \left(\frac{k'}{\omega \mu_o}\right) e^{-2k''z}$$

Time average power per unit area decays exponentially with distance because energy is dissipated in a conductive medium due to I^2R (or $J.E$) type of losses and this energy dissipated is taken away from the plane wave

Loss Tangent and Dielectric Relaxation Time - I

The complex wavevector is: $k = \omega \dfrac{n_{eff}(\omega)}{c}$

The complex refractive index is: $n_{eff}(\omega) = \sqrt{\dfrac{\varepsilon}{\varepsilon_o}}\sqrt{1 - j\underbrace{\dfrac{\sigma}{\omega\varepsilon}}_{\text{Loss tangent}}}$

Loss tangent = $\dfrac{\sigma}{\omega\varepsilon}$

But the dielectric relaxation time was: $\tau_d = \dfrac{\varepsilon}{\sigma}$

$\Rightarrow$ **Loss tangent =** $\dfrac{1}{\omega\tau_d}$ **And:** $n_{eff}(\omega) = \sqrt{\dfrac{\varepsilon}{\varepsilon_o}}\sqrt{1 - j\underbrace{\dfrac{1}{\omega\tau_d}}_{\text{Loss tangent}}}$

Loss Tangent and Dielectric Relaxation Time - II

There are two possible scenarios:

High frequency and/or low conductivity case (e.g. lossy dielectrics)

$\omega\tau_d >> 1$ or $\dfrac{\omega\varepsilon}{\sigma} >> 1$

The frequency is much greater than the inverse dielectric relaxation time

$\Rightarrow$The conductive medium does not have enough time to react to the electromagnetic wave
$\Rightarrow$No appreciable currents flow in the conductive medium

Low frequency and/or high conductivity case (e.g. Imperfect metals)

$\omega\tau_d << 1$ or $\dfrac{\omega\varepsilon}{\sigma} << 1$

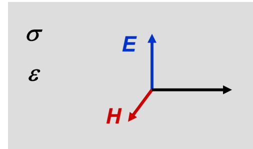
The frequency is much smaller than the inverse dielectric relaxation time

$\Rightarrow$The conductive medium has enough time to react to the electromagnetic wave
$\Rightarrow$Appreciable currents flow in the conductive medium

Waves in a Conductive Medium – Lossy Dielectrics

Plane waves:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$



Dispersion relation

$$k = \omega\, \frac{n_{eff}(\omega)}{c}$$

Refractive index

$$n_{eff}(\omega) = \sqrt{\frac{\varepsilon}{\varepsilon_o}}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

$$\boxed{\frac{\sigma}{\omega\varepsilon} = \frac{1}{\omega\tau_d} = \text{loss tangent}}$$

Lossy dielectric approximation:

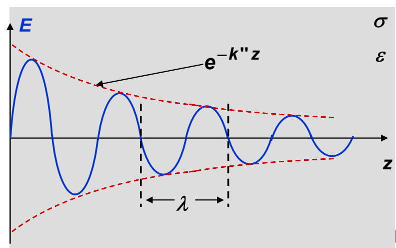
If $\dfrac{\sigma}{\omega\varepsilon} << 1$ then: $n_{eff}(\omega) \approx \sqrt{\dfrac{\varepsilon}{\varepsilon_o}}\left(1 - j\dfrac{\sigma}{2\omega\varepsilon}\right)$

$$\Rightarrow \quad k = k' - j k'' = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_o}}\left(1 - j\frac{\sigma}{2\,\omega\varepsilon}\right)$$

$$k' = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_o}} \qquad k'' = \frac{\sigma}{2}\sqrt{\frac{\mu_o}{\varepsilon}} \qquad \lambda = \frac{2\pi}{k'}$$

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k' z}\, e^{-k'' z}$$



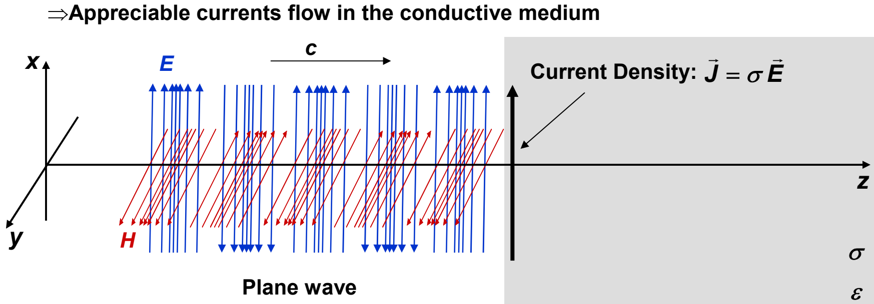
Waves in a Conductive Medium – Imperfect Metals

Now consider the case when:

$$\omega\tau_d << 1 \qquad \text{or} \qquad \frac{\omega\varepsilon}{\sigma} << 1$$

The frequency is much smaller than the inverse dielectric relaxation time

⇒The conductive medium has enough time to react to the electromagnetic wave
⇒Appreciable currents flow in the conductive medium

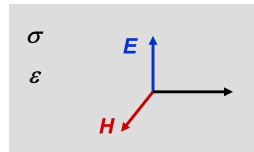


These currents try to screen out the magnetic field and, therefore, prevent the electromagnetic wave from going into the conductor

Waves in a Conductive Medium – Imperfect Metals

Plane waves:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$



Dispersion relation

$$k = \omega \frac{n_{eff}(\omega)}{c}$$

Refractive index

$$n_{eff}(\omega) = \sqrt{\frac{\varepsilon}{\varepsilon_o}} \sqrt{1 - j\frac{\sigma}{\omega \varepsilon}}$$

$$\boxed{\frac{\sigma}{\omega \varepsilon} = \frac{1}{\omega \tau_d} = \text{loss tangent}}$$

Imperfect metal approximation:

Suppose $\dfrac{\sigma}{\omega \varepsilon} >> 1$ then: $n_{eff}(\omega) \approx \sqrt{\dfrac{\varepsilon}{\varepsilon_o}} \sqrt{-j\dfrac{\sigma}{\omega \varepsilon}} = \sqrt{\dfrac{\sigma}{2\omega \varepsilon_o}}(1-j)$

$$\Rightarrow \quad k = k' - j k'' = \sqrt{\frac{\sigma \omega \mu_o}{2}}(1-j) = \frac{1}{\delta}(1-j)$$

$$k' = \sqrt{\frac{\omega \mu_o \sigma}{2}} = \frac{1}{\delta} \qquad k'' = \sqrt{\frac{\omega \mu_o \sigma}{2}} = \frac{1}{\delta} \qquad \lambda = \frac{2\pi}{k'} = 2\pi\,\delta$$

$\boxed{\delta = \text{penetration depth or skin-depth}}$

Waves in a Conductive Medium – Imperfect Metals

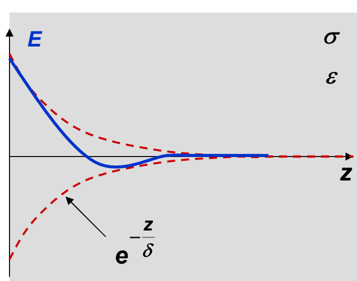
Due to current screening the wave decays within a few skin-depths:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k' z}\, e^{-k'' z}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k' z}\, e^{-\frac{z}{\delta}}$$

$\boxed{\delta = \text{penetration depth or skin-depth}}$

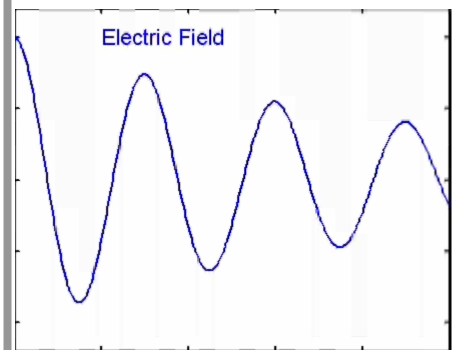


Since the wavelength λ inside the medium is $2\pi\delta$, the wave hardly propagates one wavelength distance into the medium

The screening current density, given by $\vec{J} = \sigma \vec{E}$, is non-zero only in a layer of thickness equal to skin-depth δ near the surface

Waves in a Conductive Medium

Lossy Dielectrics



Imperfect Metals