Lecture 14

Time Harmonic Fields

In this lecture you will learn:

- Complex mathematics for time-harmonic fields
- Maxwell's equations for time-harmonic fields
- Complex Poynting vector

Time-Harmonic Fields

E and H-fields for a plane wave are (from last lecture):

$$\vec{E}(\vec{r},t) = \hat{n}\, E_o \cos\left(\omega t - \vec{k}.\vec{r}\right)$$

$$\vec{H}(\vec{r},t) = \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\cos\left(\omega t - \vec{k}.\vec{r}\right)$$

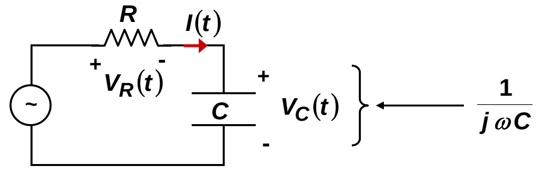
$$\left\{\begin{array}{l} \vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \vec{k}.\hat{n} = 0 \\ \omega = k\,c \end{array}\right.$$

- Fields for which the time variation is sinusoidal are called time-harmonic fields.
- Plane waves are just one example of time-harmonic fields
- In the rest of this course, 95% of the material will deal with time-harmonic fields

Time-Harmonic Signals in Circuits – Sinusoidal Steady State

Consider an RC circuit driven by a sinusoidal voltage source:

$V(t) = V_o \cos(\omega t)$



Remember phasors from ECE210…..

$$V(t) = \text{Re}\left[V_o\, e^{j\omega t}\right] \qquad V_R(t) = \text{Re}\left[V_R(\omega) e^{j\omega t}\right] \qquad I(t) = \text{Re}\left[I(\omega)\, e^{j\omega t}\right]$$

$$\Rightarrow \qquad V_R(\omega) = V_o \frac{j\omega RC}{1 + j\omega RC} \qquad I(\omega) = V_o \frac{j\omega C}{1 + j\omega RC}$$

Time-average power dissipation in the resistor:

$$\langle V_R(t) I(t) \rangle = \frac{1}{2} \text{Re}\left[V_R(\omega)\, I^*(\omega)\right] = \frac{V_o^2}{2R}\left[\frac{(\omega RC)^2}{1 + (\omega RC)^2}\right]$$

Time-Harmonic Fields and Complex Notation

Basic idea:

If the time-variation of fields is known a-priori to be sinusoidal (i.e. the fields are known to be time-harmonic) then, in order to simplify the math, one may not carry around the time dependence explicitly in calculations

Lets look at plane waves as an example to see how the complex notation can be used to factor out the sinusoidal time dependence

Some useful trigonometric Identities to recall before we start:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

Expression for the E-field of a plane wave in complex notation:

$$\vec{E}(\vec{r},t) = \hat{n} E_o \cos\left(\omega t - \vec{k}.\vec{r}\right)$$

$$\vec{E}(\vec{r},t) = \hat{n} E_o \frac{e^{j\left(\omega t - \vec{k}.\vec{r}\right)} + e^{-j\left(\omega t - \vec{k}.\vec{r}\right)}}{2} = \text{Re}\left[\hat{n} E_o e^{j\left(\omega t - \vec{k}.\vec{r}\right)}\right]$$

Time-Harmonic Fields and Vector Phasors

For the E-field of a plane wave we had…

$$\vec{E}(\vec{r},t) = \hat{n}\, E_o \cos\left(\omega t - \vec{k}.\vec{r}\right) \quad \Rightarrow \quad \vec{E}(\vec{r},t) = \text{Re}\left[\hat{n}\; E_o\; e^{j\left(\omega t - \vec{k}.\vec{r}\right)}\right]$$

Do a little more manipulation …

$$\vec{E}(\vec{r},t) = \text{Re}\left[\hat{n}\; E_o\; e^{j\left(\omega t - \vec{k}.\vec{r}\right)}\right]$$

$$= \text{Re}\left[\hat{n}\; E_o\; e^{-j\,\vec{k}.\vec{r}}\; e^{j\,\omega t}\right]$$

$$\vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\; e^{j\,\omega t}\right] \longrightarrow \left\{ \text{where: } \vec{E}(\vec{r}) = \hat{n}\; E_o\; e^{-j\,\vec{k}.\vec{r}} \right.$$

The quantity $\vec{E}(\vec{r})$, which is a time-independent complex vector, is a vector phasor for the plane wave

In the book, the vector phasor has an additional under-line and written as: $\underline{\vec{E}}(\vec{r})$

Complex Notation

For the E-field of a plane wave we had…

$$\vec{E}(\vec{r},t) = \hat{n}\, E_o \cos\left(\omega t - \vec{k}.\vec{r}\right) \quad \Rightarrow \quad \vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\; e^{j\,\omega t}\right]$$

Now generalize to all time-harmonic fields:

- All time-harmonic fields (not just plane waves) can be written in the form:

$$\vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\; e^{j\,\omega t}\right]$$

where $\vec{E}(\vec{r})$ is a complex time-independent vector phasor

- Given a vector phasor $\vec{E}(\vec{r})$ for a time-harmonic field, one can find the actual time-dependent field as follows:

$$\vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\; e^{j\,\omega t}\right]$$

Example: Suppose I give you the following vector phasor for a plane wave:

$$\vec{E}(\vec{r}) = \hat{x}\;\; A\, e^{-j\,k\,z}$$

Then you can find the actual time-dependent E-field as follows:

$$\vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\; e^{j\,\omega t}\right] = \text{Re}\left[\hat{x}\;\; A\, e^{-j\,k\,z}\;\; e^{j\,\omega t}\right]$$

$$= \hat{x}\, A \cos\left(\omega t - k\,z\right)$$

Maxwell's Equations for Phasors - I

Let the time-harmonic E and H-fields be:

$$\vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\, e^{j\omega t}\right] \qquad \vec{H}(\vec{r},t) = \text{Re}\left[\vec{H}(\vec{r})\, e^{j\omega t}\right]$$

Assume that the time-variations of charge density and current density are also sinusoidal:

$$\rho(\vec{r},t) = \text{Re}\left[\rho(\vec{r})\, e^{j\omega t}\right] \qquad \vec{J}(\vec{r},t) = \text{Re}\left[\vec{J}(\vec{r})\, e^{j\omega t}\right]$$

Now we substitute these expressions in Maxwell's equations one by one

Gauss' Law:

$$\nabla . \varepsilon_o \vec{E}(\vec{r},t) = \rho(\vec{r},t)$$

$$\Rightarrow \quad \text{Re}\left[\nabla . \varepsilon_o \vec{E}(\vec{r})\, e^{j\omega t}\right] = \text{Re}\left[\rho(\vec{r})\, e^{j\omega t}\right]$$

The only way the above can be true for all time is if:

$$\nabla . \varepsilon_o \vec{E}(\vec{r}) = \rho(\vec{r}) \qquad \text{——— (1)}$$

Gauss' Law for the Magnetic Field:

$$\nabla . \mu_o \vec{H}(\vec{r},t) = 0 \quad \Rightarrow \quad \nabla . \mu_o \vec{H}(\vec{r}) = 0 \qquad \text{——— (2)}$$

ECE 303 – Fall 2007 – Farhan Rana – Cornell University

Maxwell's Equations for Phasors - II

The time-harmonic E and H-fields are:

$$\vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\, e^{j\omega t}\right] \qquad \vec{H}(\vec{r},t) = \text{Re}\left[\vec{H}(\vec{r})\, e^{j\omega t}\right]$$

$$\rho(\vec{r},t) = \text{Re}\left[\rho(\vec{r})\, e^{j\omega t}\right] \qquad \vec{J}(\vec{r},t) = \text{Re}\left[\vec{J}(\vec{r})\, e^{j\omega t}\right]$$

Faraday's Law:

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \mu_o \vec{H}(\vec{r},t)}{\partial t}$$

$$\Rightarrow \quad \text{Re}\left[\nabla \times \vec{E}(\vec{r})\, e^{j\omega t}\right] = \text{Re}\left[-j\omega \mu_o \vec{H}(\vec{r})\, e^{j\omega t}\right]$$

The only way the above can be true for all time is if:

$$\nabla \times \vec{E}(\vec{r}) = -j\omega \mu_o \vec{H}(\vec{r}) \qquad \text{——— (3)}$$

Ampere's Law:

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \varepsilon_o \vec{E}(\vec{r},t)}{\partial t}$$

$$\Rightarrow \quad \text{Re}\left[\nabla \times \vec{H}(\vec{r})\, e^{j\omega t}\right] = \text{Re}\left[\vec{J}(\vec{r})\, e^{j\omega t} + j\omega \varepsilon_o \vec{E}(\vec{r})\, e^{j\omega t}\right]$$

The only way the above can be true for all time is if:

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega \varepsilon_o \vec{E}(\vec{r}) \qquad \text{——— (4)}$$

ECE 303 – Fall 2007 – Farhan Rana – Cornell University

Maxwell's Equations for Phasors - III

Let the time-harmonic E and H-fields be given as:

$$\vec{E}(\vec{r},t)=\text{Re}\left[\vec{E}(\vec{r})\,e^{j\omega t}\right] \qquad \vec{H}(\vec{r},t)=\text{Re}\left[\vec{H}(\vec{r})\,e^{j\omega t}\right]$$

$$\rho(\vec{r},t)=\text{Re}\left[\rho(\vec{r})\,e^{j\omega t}\right] \qquad \vec{J}(\vec{r},t)=\text{Re}\left[\vec{J}(\vec{r})\,e^{j\omega t}\right]$$

Maxwell's equations for the vector phasors of time-harmonic fields are then:

Gauss' Law:

$$\nabla.\varepsilon_o\vec{E}(\vec{r})=\rho(\vec{r})$$

Gauss' Law for the Magnetic Field:

$$\nabla.\mu_o\vec{H}(\vec{r})=0$$

Faraday's Law:

$$\nabla\times\vec{E}(\vec{r})=-j\,\omega\,\mu_o\,\vec{H}(\vec{r})$$

Ampere's Law:

$$\nabla\times\vec{H}(\vec{r})=\vec{J}(\vec{r})+j\,\omega\,\varepsilon_o\,\vec{E}(\vec{r})$$

Calculations in the Complex Notation

Suppose for a plane wave we know the E-field to be:

$$\vec{E}(\vec{r},t)=\text{Re}\left[\vec{E}(\vec{r})\,e^{j\omega t}\right] \qquad \text{where} \qquad \vec{E}(\vec{r})=\hat{n}\,E_o\,e^{-j\vec{k}.\vec{r}}$$

How does one find the vector phasor for the H-field?

$$\vec{H}(\vec{r},t)=\text{Re}\left[\vec{H}(\vec{r})\,e^{j\omega t}\right]$$

Use Faraday's law for time-harmonic fields:

$$\nabla\times\vec{E}(\vec{r})=-j\,\omega\,\mu_o\,\vec{H}(\vec{r})$$

$$\Rightarrow\quad \vec{H}(\vec{r})=\frac{j}{\omega\,\mu_o}\nabla\times\vec{E}(\vec{r})$$

$$\Rightarrow\quad \vec{H}(\vec{r})=\frac{j}{\omega\,\mu_o}\nabla\times\left(\hat{n}\,E_o\,e^{-j\vec{k}.\vec{r}}\right)$$

$$\Rightarrow\quad \vec{H}(\vec{r})=\frac{j}{\omega\,\mu_o}\left(-j\,\vec{k}\times\hat{n}\right)E_o\,e^{-j\vec{k}.\vec{r}}$$

$$\Rightarrow\quad \vec{H}(\vec{r})=\frac{k}{\omega\,\mu_o}\left(\hat{k}\times\hat{n}\right)E_o\,e^{-j\vec{k}.\vec{r}} \quad\longrightarrow\quad \left\{\ \vec{k}=k\,\hat{k}\right.$$

$$\Rightarrow\quad \vec{H}(\vec{r})=\left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\,e^{-j\vec{k}.\vec{r}}$$

Complex Poynting Vector

Suppose for a plane wave we know the E-field and H-field phasors to be:

$$\vec{E}(\vec{r}) = \hat{n}\, E_o\, e^{-j\vec{k}.\vec{r}}$$

$$\vec{H}(\vec{r}) = \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\, e^{-j\vec{k}.\vec{r}}$$

How does one find the time-average power per unit area carried by the wave?

Define a complex Poynting vector as:

$$\boxed{\vec{S}(\vec{r}) = \vec{E}(\vec{r})\times\vec{H}^*(\vec{r})}$$

Claim: The time-average power per unit area is one-half of the real part of the complex Poynting vector

Check:

$$\left\langle\vec{S}(\vec{r},t)\right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\hat{n}\times\left(\hat{k}\times\hat{n}\right)\frac{E_o^2}{\eta_o}\right] = \hat{k}\,\frac{E_o^2}{2\,\eta_o}$$

→ which is indeed the right answer

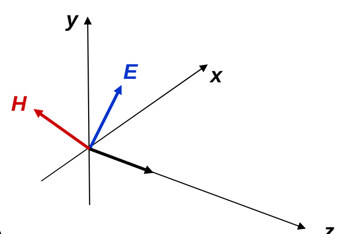
More Calculations in the Complex Notation - I

Example:

Consider a plane wave with E-field of amplitude E_o and pointing in a direction 45-degrees w.r.t. the x-axis (as shown) and traveling in the +z-direction

Write expression for the E-field phasor:

$$\vec{E}(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o\, e^{-j\,k\,z}$$



Write expression for the H-field phasor:

$$\vec{H}(\vec{r}) = \frac{j}{\omega\,\mu_o}\nabla\times\vec{E}(\vec{r})$$

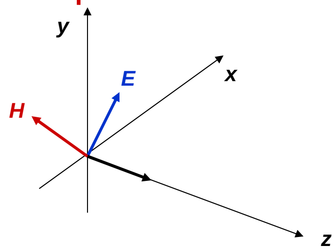
$$\Rightarrow\quad \vec{H}(\vec{r}) = \frac{j}{\omega\,\mu_o}\nabla\times\left(\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o\, e^{-j\,k\,z}\right)$$

$$\Rightarrow\quad \vec{H}(\vec{r}) = \frac{j}{\omega\,\mu_o}\left(-j\,k\,\hat{z}\times\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)\right) E_o\, e^{-j\,k\,z} \quad\longrightarrow\quad \left\{\vec{k} = k\,\hat{z}\right.$$

$$\Rightarrow\quad \vec{H}(\vec{r}) = \frac{k}{\omega\,\mu_o}\left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right) E_o\, e^{-j\,k\,z}$$

$$\Rightarrow\quad \vec{H}(\vec{r}) = \left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right)\frac{E_o}{\eta_o}\, e^{-j\,k\,z}$$

More Calculations in the Complex Notation - II



The E-field phasor is:

$$\vec{E}(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o \, e^{-j\,k\,z}$$

The H-field phasor is:

$$\vec{H}(\vec{r}) = \left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right) \frac{E_o}{\eta_o} \, e^{-j\,k\,z}$$

Find the complex Poynting vector:

$$\vec{S}(\vec{r}) = \vec{E}(\vec{r}) \times \vec{H}^{*}(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) \times \left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right) \frac{E_o^2}{\eta_o}$$

$$= \hat{z}\,\frac{E_o^2}{\eta_o}$$

Find the time-average power per unit area:

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right] = \hat{z}\,\frac{E_o^2}{2\,\eta_o}$$

ECE 303 – Fall 2007 – Farhan Rana – Cornell University