Lecture 13

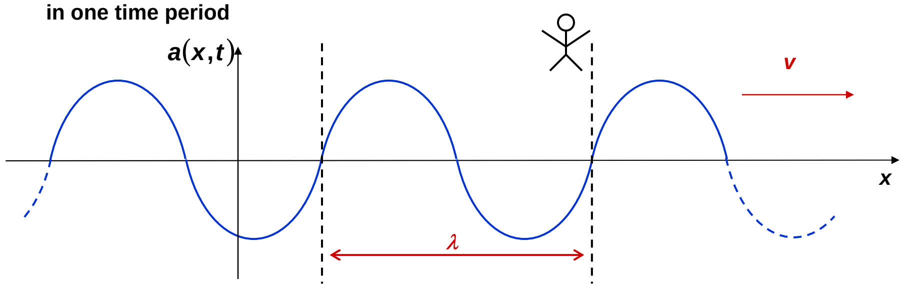
Electromagnetic Waves in Free Space

In this lecture you will learn:

- Electromagnetic wave equation in free space
- Uniform plane wave solutions of the wave equation
- Energy and power of electromagnetic waves

Basic Wave Motion

Consider a wave moving in the +*x*-direction:
The wave travels a distance equal to one wavelength in one time period



v = velocity of wave propagation
λ = wavelength of the wave
f = frequency of the wave
T = period = $1/f$

$$v = \text{velocity} = \frac{\text{distance}}{\text{time}} = \frac{\lambda}{T} = f\lambda$$

Basic relation for wave motion: $\boxed{f\,\lambda = v}$

1-D wave equation: $$\frac{\partial^2 a(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 a(x,t)}{\partial t^2}$$

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Electromagnetic Wave Motion - I

$$\nabla . \varepsilon_o \vec{E} = \rho \qquad \nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$

$$\nabla . \mu_o \vec{H} = 0 \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \varepsilon_o \vec{E}}{\partial t}$$

Time varying electric and magnetic fields are coupled - this coupling is responsible for the propagation of electromagnetic waves

Electromagnetic Wave Equation:

Assume free space: $\Rightarrow \quad \rho = \vec{J} = 0$

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla \times \left(\frac{\partial \mu_o \vec{H}}{\partial t}\right) = -\frac{\partial \mu_o \nabla \times \vec{H}}{\partial t} = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \nabla \times \left(\nabla \times \vec{E}\right) = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \approx 3 \times 10^8 \text{ m/s}$$

$$\Rightarrow \quad \nabla \times \left(\nabla \times \vec{E}\right) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Electromagnetic Wave Motion - II

$$\Rightarrow \quad \nabla \times \left(\nabla \times \vec{E}\right) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Use the vector Identity: $\nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla . \vec{F}\right) - \nabla^2 \vec{F}$

$$\Rightarrow \quad \nabla\left(\cancelto{0}{\nabla . \vec{E}}\right) - \nabla^2 \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow \left\{ \nabla . \varepsilon_o \vec{E} = \rho = 0 \right.$$

$$\Rightarrow \quad \boxed{\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}} \longrightarrow$$

Equation for a wave traveling at speed *c* in free space

$$(1) \quad \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}$$

$$(2) \quad \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}$$

$$(3) \quad \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}$$

Wave equation is essentially three equations stacked together – one for each component of the E-field

Wave must also satisfy: $\nabla . \varepsilon_o \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$

Electromagnetic Wave Motion - III

The H-field also satisfies a similar wave equation

Start from Maxwell's equations: $\nabla\times\vec{E} = -\dfrac{\partial\,\mu_o\vec{H}}{\partial t}$ $\qquad$ $\nabla\times\vec{H} = \vec{J} + \dfrac{\partial\,\varepsilon_o\,\vec{E}}{\partial t}$

Assume free space: $\Rightarrow \quad \rho = \vec{J} = 0$

$$\nabla\times\left(\nabla\times\vec{H}\right) = \nabla\times\left(\frac{\partial\,\varepsilon_o\vec{E}}{\partial t}\right) = -\frac{\partial\,\varepsilon_o\,\nabla\times\vec{E}}{\partial t} = -\mu_o\,\varepsilon_o\,\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\Rightarrow\quad \nabla\times\left(\nabla\times\vec{H}\right) = -\mu_o\,\varepsilon_o\,\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\Rightarrow\quad \nabla\times\left(\nabla\times\vec{H}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\Rightarrow\quad \nabla\left(\cancelto{0}{\nabla.\vec{H}}\right) - \nabla^2\vec{H} = -\frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} \longrightarrow \left\{\ \nabla.\,\mu_o\,\vec{H} = 0\right.$$

$$\Rightarrow\quad \nabla^2\vec{H} = \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}$$

General Solutions of Electromagnetic Wave Equation

Assume only x-component of the E-field: $\vec{E} = \hat{x}\,E_x$

$\nabla.\,\varepsilon_o\,\vec{E} = 0 \quad\Rightarrow\quad \dfrac{\partial E_x}{\partial x} = 0 \longrightarrow \Big\{$ **The *x*-component cannot have *x*-dependence**

So assume: $\vec{E} = \hat{x}\,E_x(z,t)$

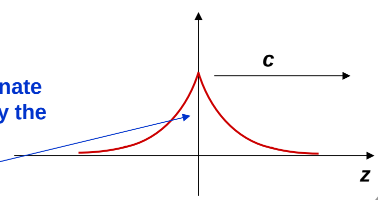
And plug it into the wave equation: $\nabla^2\vec{E} = \dfrac{1}{c^2}\dfrac{\partial^2\vec{E}}{\partial t^2}$

To obtain: $\dfrac{\partial^2 E_x(z,t)}{\partial z^2} = \dfrac{1}{c^2}\dfrac{\partial^2 E_x(z,t)}{\partial t^2}$

Solution is: $E_x(z,t) = g(z \pm c\,t)$

Any function *g* whose dependence on co-ordinate *z* and time *t* is in the form of ($z \pm ct$) will satisfy the above equation

Example: $E_x(z,t) = E_o\exp\left(-\alpha\left|z - c\,t\right|\right)$



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Sinusoidal Solutions of Electromagnetic Wave Equation - I

$$\frac{\partial^2 E_x(z,t)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x(z,t)}{\partial t^2} \qquad \Rightarrow \qquad \vec{E} = \hat{x}\,E_x(z-ct)$$

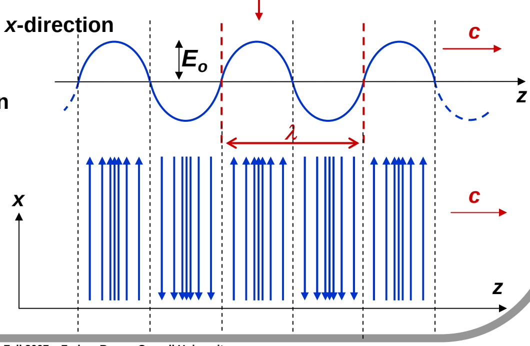
The most commonly used solutions are sinusoids, for example:

$$\vec{E} = \hat{x}\,E_x(z-ct) = \hat{x}\,E_o\cos\left(\frac{2\pi}{\lambda}(z-c\,t)\right)$$

This solution represents a wave that:

i) Has electric field pointing in x-direction
ii) Has wavelength λ
iii) Has frequency $f = c/\lambda$
iv) Is moving in the $+z$-direction

In space the electric field looks as shown here (remember that the density of field lines correspond to the strength of the E-field)



Sinusoidal Solutions of Electromagnetic Wave Equation - II

The sinusoidal solution,

$$\vec{E} = \hat{x}\,E_x(z-ct) = \hat{x}\,E_o\cos\left(\frac{2\pi}{\lambda}(z-c\,t)\right)$$

can also be written as:

$$\vec{E} = \hat{x}\,E_o\cos\left(\frac{2\pi\,c}{\lambda}\left(t-\frac{z}{c}\right)\right) = \hat{x}\,E_o\cos\left(\frac{2\pi\,c}{\lambda}t - \frac{2\pi}{\lambda}z\right)$$

Define:

$$\omega = \frac{2\pi\,c}{\lambda} = 2\pi f \qquad \text{and} \qquad k = \frac{2\pi}{\lambda}$$

To get:

$$\vec{E} = \hat{x}\,E_o\cos(\omega t - k\,z)$$

ω = angular frequency (units: radians/sec)
k = wave-vector (units: 1/m)

Note that:

$$f\,\lambda = v \quad \Rightarrow \quad \boxed{\omega = k\,c}$$

A dispersion relation

Sinusoidal Solutions of Electromagnetic Wave Equation - III

What about the magnetic field?

Recall the E and H-fields are coupled ⟶

$$\left\{ \begin{array}{l} \nabla \times \vec{E} = -\dfrac{\partial \mu_o \vec{H}}{\partial t} \\ \nabla \times \vec{H} = \dfrac{\partial \varepsilon_o \vec{E}}{\partial t} \end{array} \right.$$

So an E-field must be accompanied by an H-field which can be calculated from the equation:

$$\nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t} \qquad \left(\text{or even from } \nabla \times \vec{H} = \frac{\partial \varepsilon_o \vec{E}}{\partial t} \right)$$

Plug in the following solution for the E-field: $\vec{E} = \hat{x}\, E_o \cos(\omega t - k z)$

$$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_o} \nabla \times \vec{E} = -\hat{y}\, \frac{k}{\mu_o} E_o \sin(\omega t - k z)$$

$$\Rightarrow \quad \vec{H} = \hat{y}\, \frac{k}{\mu_o\, \omega} E_o \cos(\omega t - k z) \longrightarrow \left\{ \frac{\mu_o \omega}{k} = \mu_o\, c = \sqrt{\frac{\mu_o}{\varepsilon_o}} = \eta_o \right.$$

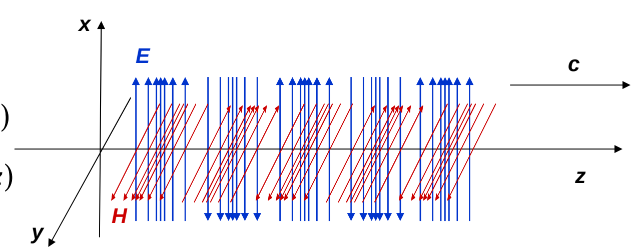
$$\Rightarrow \quad \vec{H} = \hat{y}\, \frac{E_o}{\eta_o} \cos(\omega t - k z)$$

$$\boxed{\eta_o = \text{impedance of free space} \approx 377\,\Omega}$$

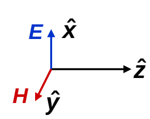
Sinusoidal Solutions of Electromagnetic Wave Equation - IV

$$\vec{E} = \hat{x}\, E_o \cos(\omega t - k z)$$

$$\vec{H} = \hat{y}\, \frac{E_o}{\eta_o} \cos(\omega t - k z)$$



Cartoon depiction:



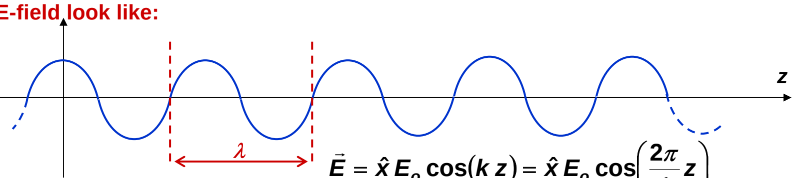
- These solutions of the wave equation are called **uniform plane waves**
- The E-field (and the H-field as well) is constant over any infinite plane that is parallel to the *x-y* plane – in more technical terms, **the surfaces of constant phase are infinite planes**
- The pattern shown above moves with a velocity equal to $\dfrac{\omega}{k} = c$

Sinusoidal Solutions of Electromagnetic Wave Equation - V

Consider the plane wave:

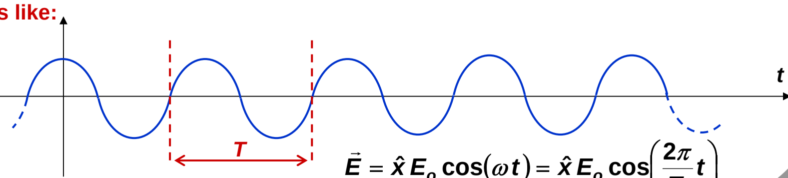
$$\vec{E} = \hat{x}\, E_o \cos(\omega t - k z) \qquad \vec{H} = \hat{y}\, \frac{E_o}{\eta_o} \cos(\omega t - k z)$$

If a person takes a snapshot of the wave in space at any time, say at $t = 0$, he will see E-field look like:



$$\vec{E} = \hat{x}\, E_o \cos(k z) = \hat{x}\, E_o \cos\left(\frac{2\pi}{\lambda} z\right)$$

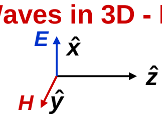
If a person sits at one location, say $z = 0$, he will see an oscillating E-field in time that looks like:



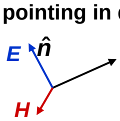
$$\vec{E} = \hat{x}\, E_o \cos(\omega t) = \hat{x}\, E_o \cos\left(\frac{2\pi}{T} t\right)$$

Plane Waves in 3D - I

So far we have found one solution:



What about plane waves with E-field pointing in direction $\hat{n}$ and traveling in some arbitrary direction in 3D ?



The answer is:

$$\vec{E}(\vec{r},t) = \hat{n}\, E_o \cos(\omega t - \vec{k}.\vec{r}) \longrightarrow \begin{cases} \vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \end{cases}$$

Lets see if this solution satisfies the wave equation:

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad -\vec{k}.\vec{k}\left[\hat{n}\, E_o \cos(\omega t - \vec{k}.\vec{r})\right] = -\frac{\omega^2}{c^2}\left[\hat{n}\, E_o \cos(\omega t - \vec{k}.\vec{r})\right]$$

$$\Rightarrow \quad \omega^2 = (\vec{k}.\vec{k})\, c^2 = k^2 c^2$$

$$\Rightarrow \quad \boxed{\omega = k\, c} \longrightarrow$$ **The solution can only be correct if: $\omega = k\, c$**

Plane Waves in 3D - II

The solution for a plane wave in 3D is:

$$\vec{E}(\vec{r},t) = \hat{n}\, E_o \cos(\omega t - \vec{k}.\vec{r}) \longrightarrow \begin{cases} \vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \\ \omega = k\,c \end{cases}$$

The solution must also satisfy:

$$\nabla.\, \varepsilon_o\, \vec{E} = 0$$

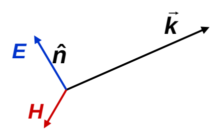
Plug in the solution to check:

$$\nabla.\, \varepsilon_o\, \vec{E} = \varepsilon_o\, \nabla.\left[\hat{n}\, E_o \cos(\omega t - \vec{k}.\vec{r})\right] = \varepsilon_o\, (\vec{k}.\hat{n})\, E_o \cos(\omega t - \vec{k}.\vec{r})$$

$$\nabla.\, \varepsilon_o\, \vec{E} = 0 \quad \text{provided} \quad \boxed{\vec{k}.\hat{n} = 0}$$

The solution can only be correct if the unit vector $\hat{n}$ is perpendicular to $\vec{k}$

The solution in fact corresponds to a plane wave traveling in the direction: $\vec{k}$
With E-field pointing in the direction: $\hat{n}$

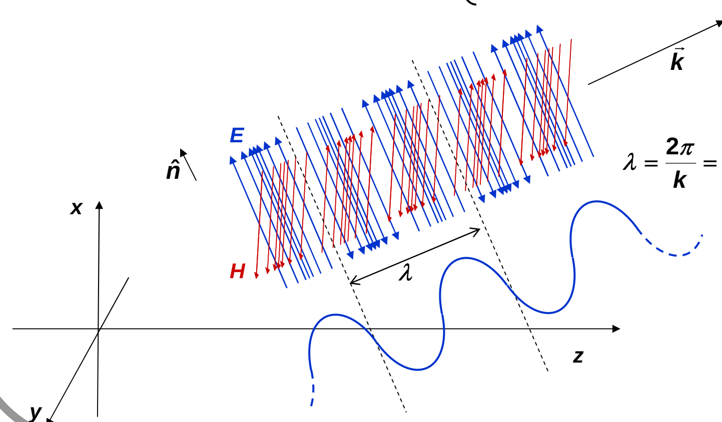


$\vec{k}.\hat{n} = 0 \quad \Rightarrow$ E-field is perpendicular to the direction of travel

Plane Waves in 3D - III

The solution for a plane wave in 3D is:

$$\vec{E}(\vec{r},t) = \hat{n}\, E_o \cos(\omega t - \vec{k}.\vec{r}) \longrightarrow \begin{cases} \vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \omega = k\,c \\ \vec{k}.\hat{n} = 0 \end{cases}$$



$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{k_x^2 + k_y^2 + k_z^2}}$$

Plane Waves in 3D - IV

If the E-field for a plane wave solution in 3D is:

$$\vec{E}(\vec{r},t) = \hat{n}\,E_o \cos(\omega t - \vec{k}.\vec{r})$$

Then what is the magnetic field?

$$\left\{ \begin{array}{l} \vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \omega = k\,c \\ \vec{k}.\hat{n} = 0 \end{array} \right.$$

Use the same old equation:

$$\nabla \times \vec{E} = -\frac{\partial\,\mu_o \vec{H}}{\partial t} \qquad \left(\text{or even from } \nabla \times \vec{H} = \frac{\partial\,\varepsilon_o \vec{E}}{\partial t} \right)$$

Plug in the following solution for the E-field: $\vec{E}(\vec{r},t) = \hat{n}\,E_o \cos(\omega t - \vec{k}.\vec{r})$

$$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_o}\nabla \times \vec{E} = -\frac{(\vec{k}\times\hat{n})}{\mu_o} E_o \sin(\omega t - \vec{k}.\vec{r})$$

$$\Rightarrow \quad \vec{H} = \frac{(\vec{k}\times\hat{n})}{\mu_o\,\omega} E_o \cos(\omega t - \vec{k}.\vec{r}) \longrightarrow \left\{ \vec{k} = k\,\hat{k} \right.$$

$$\Rightarrow \quad \vec{H} = (\hat{k}\times\hat{n})\frac{E_o}{\eta_o} \cos(\omega t - \vec{k}.\vec{r}) \longrightarrow$$

Direction of H-field given by $\hat{k}\times\hat{n}$ is also perpendicular to the direction of travel

Plane Waves in 3D - V

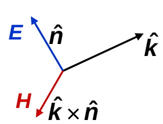
E-field for a plane wave solution in 3D:

$$\vec{E}(\vec{r},t) = \hat{n}\,E_o \cos(\omega t - \vec{k}.\vec{r})$$

H-field for a plane wave solution in 3D:

$$\vec{H}(\vec{r},t) = (\hat{k}\times\hat{n})\frac{E_o}{\eta_o} \cos(\omega t - \vec{k}.\vec{r})$$

$$\left\{ \begin{array}{l} \vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \omega = k\,c \\ \vec{k}.\hat{n} = 0 \end{array} \right.$$



Questions:

i) What about energy and power?
ii) How much power per unit area does a plane wave carry?
iii) What is the energy density of a plane wave?

Try using the Poynting vector: $\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$ (units: Joules/(m^2-sec) or: Watts/m^2)

Power Carried By Plane Waves

E-field and H-field for a plane wave solution in 3D:

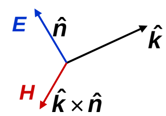
$$\vec{E}(\vec{r},t) = \hat{n}\,E_o \cos\left(\omega t - \vec{k}.\vec{r}\right) \qquad \vec{H}(\vec{r},t) = \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\cos\left(\omega t - \vec{k}.\vec{r}\right)$$

Poynting vector for a plane wave:

$$\begin{aligned}\vec{S}(\vec{r},t) &= \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\\ &= \hat{n}\times\left(\hat{k}\times\hat{n}\right)\frac{E_o^2}{\eta_o}\cos^2\left(\omega t - \vec{k}.\vec{r}\right)\\ &= \hat{k}\ \frac{E_o^2}{\eta_o}\cos^2\left(\omega t - \vec{k}.\vec{r}\right)\end{aligned}$$

Use the vector identity:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}.\vec{C}\right) - \vec{C}\left(\vec{A}.\vec{B}\right)$$



Power flows in the direction of the wave-vector

Time average power per unit area of a plane wave:

Usually one is interested in the time-average power per unit area:

$$\left\langle\vec{S}(\vec{r},t)\right\rangle = \hat{k}\ \frac{E_o^2}{\eta_o}\left\langle\cos^2\left(\omega t - \vec{k}.\vec{r}\right)\right\rangle = \hat{k}\ \frac{E_o^2}{2\,\eta_o}$$

$$\begin{aligned}&\left\langle\cos^2(\omega t + \theta)\right\rangle\\ &= \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega}\cos^2(\omega t + \theta)\,dt\\ &= \frac{1}{2}\end{aligned}$$

Power and Energy Density of Plane Waves

E-field and H-field for a plane wave solution in 3D:

$$\vec{E}(\vec{r},t) = \hat{n}\,E_o \cos\left(\omega t - \vec{k}.\vec{r}\right) \qquad \vec{H}(\vec{r},t) = \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\cos\left(\omega t - \vec{k}.\vec{r}\right)$$

Electric field energy density $= W_e(\vec{r},t) = \frac{1}{2}\varepsilon_o\,\vec{E}(\vec{r},t).\vec{E}(\vec{r},t) = \frac{1}{2}\varepsilon_o\,E_o^2\cos^2\left(\omega t - \vec{k}.\vec{r}\right)$

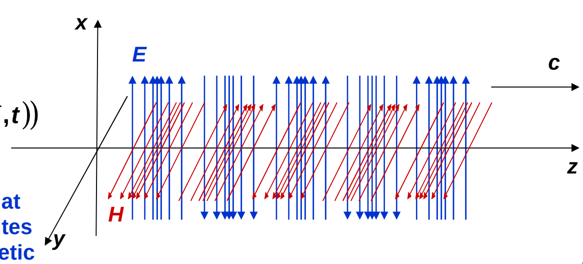
Magnetic field energy density $= W_m(\vec{r},t) = \frac{1}{2}\mu_o\,\vec{H}(\vec{r},t).\vec{H}(\vec{r},t) = \frac{1}{2}\mu_o\,\frac{E_o^2}{\eta_o^2}\cos^2\left(\omega t - \vec{k}.\vec{r}\right)$

Notice that:

$$W_e(\vec{r},t) = W_m(\vec{r},t)$$

$$\left|\vec{S}(\vec{r},t)\right| = c\left(W_e(\vec{r},t) + W_m(\vec{r},t)\right)$$

The electric and magnetic energy densities are moving at the speed *c* and this constitutes the power of an electromagnetic plane wave



Example: Plane Waves

If the fields for a plane wave solution in 3D is:

$$\vec{E}(\vec{r},t) = \hat{n}\,E_o \cos\left(\omega t - \vec{k}.\vec{r}\right)$$

$$\vec{H}(\vec{r},t) = \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\cos\left(\omega t - \vec{k}.\vec{r}\right)$$

$$\longrightarrow \left\{ \begin{array}{l} \vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \vec{k}.\hat{n} = 0 \\ \omega = k\,c \end{array} \right.$$

Lets try to write solution for a plane wave that:

i) Is moving in the -*x*-direction
ii) Has E-field pointing in the *y*-direction
iii) Has E-field amplitude R_o
iv) Has wavelength λ_o

(i) And (iv) imply: $\vec{k} = -k\,\hat{x}$ and $k = \dfrac{2\pi}{\lambda_o}$ **Wavevector is in the -*x*-direction**

Answer is:

$$\vec{E}(\vec{r},t) = \hat{y}\,R_o \cos\left(\omega t + k\,x\right)$$

$$\vec{H}(\vec{r},t) = -\hat{z}\,\frac{R_o}{\eta_o}\cos\left(\omega t + k\,x\right)$$