

## Lecture 11

### Faraday's Law and Electromagnetic Induction and Electromagnetic Energy and Power Flow

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In this lecture you will learn:

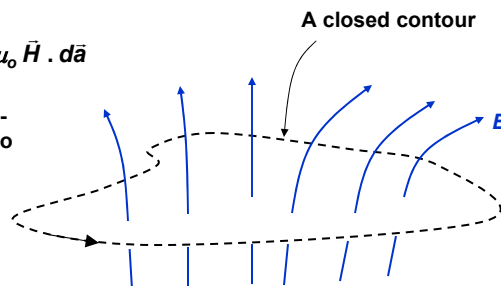
- More about Faraday's Law and Electromagnetic Induction
- The Non-uniqueness of Voltages in Magnetoquasistatics
- Electromagnetic Energy and Power Flow
- Electromagnetic Energy Stored in Capacitors and Inductors
- Appendix (some proofs)

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### Faraday's Law Revisited

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{a} = -\frac{\partial}{\partial t} \iint \mu_0 \vec{H} \cdot d\vec{a}$$

**Faraday's Law:** The line integral of E-field over a closed contour is equal to -ve of the time rate of change of the magnetic flux that goes through any arbitrary surface that is bounded by the closed contour



**Important Note:** In **electroquasistatics** the line integral of E-field over a closed contour was always zero

$$\oint \vec{E} \cdot d\vec{s} = \oint (-\nabla\phi) \cdot d\vec{s} = 0$$

In **magnetoquasistatics** this is NOT the case

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## Electromagnetic Induction and Kirchoff's Voltage Law

Kirchoff's voltage law comes from the electroquasistatic approximation:  $\oint \vec{E} \cdot d\vec{s} = 0$

$$\Rightarrow \oint \vec{E} \cdot d\vec{s} = IR_1 + IR_2 - V = 0$$

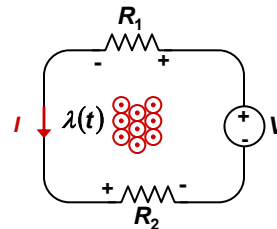
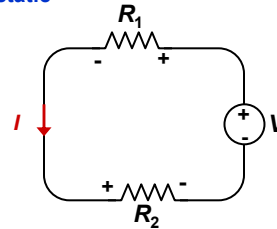
$$\Rightarrow I = \frac{V}{(R_1 + R_2)}$$

Now consider a circuit through which the magnetic flux is changing with time (Kirchoff's voltage law is violated)

$$\Rightarrow \oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{a} = -\frac{d\lambda}{dt}$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{s} = IR_1 + IR_2 - V = -\frac{d\lambda}{dt}$$

$$\Rightarrow I = \frac{V}{(R_1 + R_2)} - \frac{1}{(R_1 + R_2)} \frac{d\lambda}{dt}$$



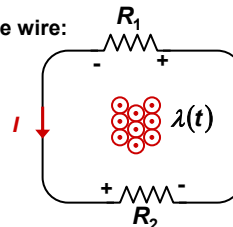
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## Lenz Law

Suppose an induced current  $I$  is flowing through the wire:

$$\Rightarrow I = -\frac{1}{(R_1 + R_2)} \frac{d\lambda}{dt}$$

The induced current in the wire produces its own magnetic field



**Lenz Law** is just an easy way to remember in which direction the induced current flows

The law states that the induced current will flow in a direction such that its own magnetic field opposes the time variation of the magnetic field that produced it

**Example:** Suppose the magnetic flux through the wire loop shown above was increasing with time (so that  $d\lambda/dt > 0$ ).

Lenz would tell us that the induced current would flow in the clockwise direction so that its own magnetic field would oppose the increasing magnetic flux through the loop

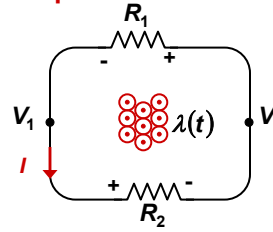
In the equation above this fact comes out from the negative sign on the right hand side

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### Non-Uniqueness of Voltages in Magnetoquasistatics - I

We have:

$$\oint \vec{E} \cdot d\vec{s} = I(R_1 + R_2) = -\frac{d\lambda}{dt}$$



**Question:** What is the voltages difference  $V_2 - V_1$  ?

One may be tempted to write .....

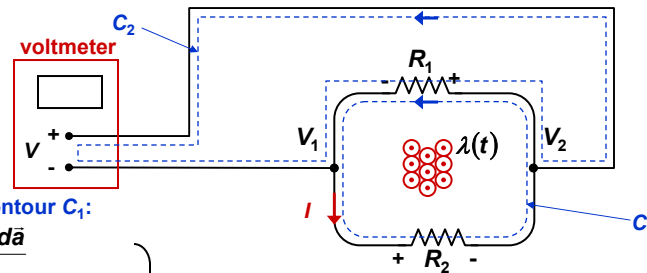
$$\left. \begin{aligned} V_1 - IR_2 &= V_2 \\ V_2 - IR_1 &= V_1 \end{aligned} \right\} \Rightarrow \begin{aligned} I &= 0 \\ V_2 - V_1 &= 0 \end{aligned}$$

..... which cannot be correct since we know that:  $I = -\frac{1}{(R_1 + R_2)} \frac{d\lambda}{dt}$

**What went wrong?** Our usual concepts of circuit theory and potentials which are based on conservative E-fields are not valid when time varying magnetic fields are present

### Non-Uniqueness of Voltages in Magnetoquasistatics - II

Lets do some real experimental measurements and see what we get for  $V_2 - V_1$



Faraday's Law for contour  $C_1$ :

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial \iint \vec{B} \cdot d\vec{a}}{\partial t}$$

$$\Rightarrow IR_1 + IR_2 = -\frac{d\lambda}{dt}$$

Faraday's Law for contour  $C_2$ :

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial \iint \vec{B} \cdot d\vec{a}}{\partial t}$$

$$\Rightarrow V - IR_1 = 0$$

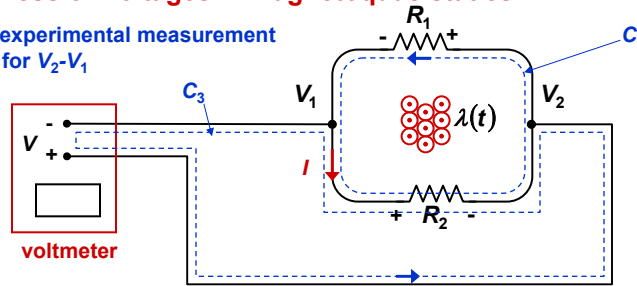
$$\Rightarrow V = V_2 - V_1 = IR_1$$

$$\Rightarrow V = V_2 - V_1 = -\frac{R_1}{R_1 + R_2} \left( \frac{d\lambda}{dt} \right)$$

This value for  $V_2 - V_1$  would actually be measured experimentally if the measurement is done as shown above

### Non-Uniqueness of Voltages in Magnetoquasistatics - III

Lets do another real experimental measurement and see what we get for  $V_2 - V_1$



Faraday's Law for contour  $C_1$ :

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial \iint \vec{B} \cdot d\vec{a}}{\partial t}$$

$$\Rightarrow IR_1 + IR_2 = -\frac{d\lambda}{dt}$$

Faraday's Law for contour  $C_3$ :

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial \iint \vec{B} \cdot d\vec{a}}{\partial t}$$

$$\Rightarrow -V - IR_2 = 0$$

$$\Rightarrow V = V_2 - V_1 = -IR_2$$

$$\Rightarrow V = V_2 - V_1 = \frac{R_2}{R_1 + R_2} \left( \frac{d\lambda}{dt} \right)$$

This value for  $V_2 - V_1$  would actually be measured experimentally if the measurement is done as shown above

**Lesson:** The result of a voltage measurement depends on how exactly you do the measurement when time varying magnetic fields are present

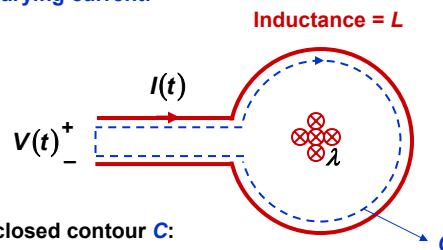
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### Inductors and Faraday's Law - I

Consider an inductor with a time varying current:

The magnetic flux is given by:

$$\lambda(t) = L I(t)$$



Consider Faraday's law over the closed contour  $C$ :

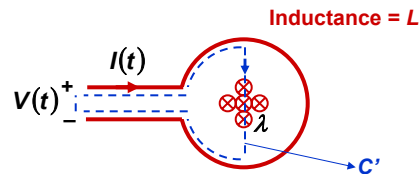
$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{a} = -\frac{\partial}{\partial t} \iint \mu_0 \vec{H} \cdot d\vec{a}$$

$$\Rightarrow V(t) = \frac{d\lambda(t)}{dt} = L \frac{dI(t)}{dt}$$

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## Inductors and Faraday's Law - II

**Question:** is there electric field anywhere else besides at the terminals??



Consider Faraday's law over the new closed contour  $C'$ :

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{a} = -\frac{\partial}{\partial t} \iint \mu_0 \vec{H} \cdot d\vec{a}$$

$$-V(t) + \int_{\text{vertical}} \vec{E} \cdot d\vec{s} = -\frac{1}{2} \frac{d\lambda(t)}{dt}$$

$$\Rightarrow \int_{\text{vertical}} \vec{E} \cdot d\vec{s} = \frac{V(t)}{2}$$

**There must be E-field inside the loop !!**

This E-field is not very easy to determine analytically.

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## The Current-Charge Continuity Equation in Electromagnetism - I

Start from:  $\nabla \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t}$

Take divergence on both sides:  $\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \frac{\partial \nabla \cdot (\epsilon \vec{E})}{\partial t}$

$$\Rightarrow 0 = \nabla \cdot \vec{J} + \frac{\partial \nabla \cdot (\epsilon \vec{E})}{\partial t}$$

Use Gauss' law:  $\nabla \cdot \epsilon \vec{E} = \rho$

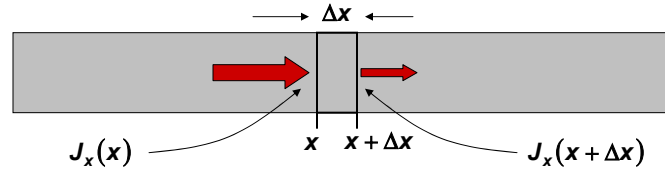
To get:  $-\nabla \cdot \vec{J} = \frac{\partial \rho}{\partial t}$   $\longrightarrow$  Current-charge continuity equation

**What does this equation mean??**

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## The Current-Charge Continuity Equation in Electromagnetism - II

Suppose the current density is spatially varying as shown below



The difference in current density at  $x$  and  $x+\Delta x$  must result in piling up of charges in the infinitesimal strip .....

$$\begin{aligned}
 J_x(x, t) - J_x(x + \Delta x, t) &= \frac{\partial \rho(x, t) \Delta x}{\partial t} \\
 \Rightarrow -\frac{J_x(x + \Delta x, t) - J_x(x, t)}{\Delta x} &= \frac{\partial \rho(x, t)}{\partial t} \\
 \Rightarrow -\frac{\partial J_x(x, t)}{\partial x} &= \frac{\partial \rho(x, t)}{\partial t}
 \end{aligned}$$

Now generalize to 3D and get the desired result:

$$-\nabla \cdot \vec{J}(\vec{r}, t) = \frac{\partial \rho(\vec{r}, t)}{\partial t}$$

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## The Current-Charge Continuity Equation in Electromagnetism - III

- A continuity equation of the form:

$$-\nabla \cdot \vec{J}(\vec{r}, t) = \frac{\partial \rho(\vec{r}, t)}{\partial t}$$

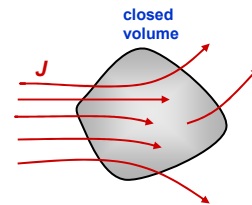
establishes a relation between the flow rate of a quantity (in the present case the flow rate of charge density) and the time rate of change of the quantity

- A continuity equation is in fact a “conservation law”

For example, the current-charge continuity equation expresses **charge conservation**. It says that charge cannot just disappear. In the integral form it becomes:

$$-\iint \vec{J}(\vec{r}, t) \cdot d\vec{a} = \frac{\partial}{\partial t} \iiint \rho(\vec{r}, t) dV = \frac{dQ(t)}{dt}$$

If there is a net inflow of charge into a closed volume then that must result in an increase in the total charge  $Q$  inside that closed volume



- In the next few slides we will try to find a **power-energy continuity equation** for the electromagnetic field

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## The Electromagnetic Power-Energy Continuity Equation - I

- We know that electric and magnetic fields have energy. Let  $W$  be the energy density (i.e. energy per unit volume) of the electromagnetic field

$$W(\vec{r}, t) = \text{Energy density (units: Joules/m}^3\text{)}$$

- Suppose we had a vector that expressed the energy flow rate (or energy flux) in Joules/(m<sup>2</sup>-sec) for the electromagnetic field

$$\vec{S}(\vec{r}, t) = \text{Energy flow rate (units: Joules/(m}^2\text{-sec))}$$

- Then the **conservation of electromagnetic energy** must be expressed in a **continuity equation** of the form:

$$-\nabla \cdot \vec{S}(\vec{r}, t) = \frac{\partial W(\vec{r}, t)}{\partial t}$$

Compare with the current-charge continuity equation:

$$-\nabla \cdot \vec{J}(\vec{r}, t) = \frac{\partial \rho(\vec{r}, t)}{\partial t}$$

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## The Electromagnetic Power-Energy Continuity Equation - II

Start from the two Maxwell's equations:

$$\nabla \times \vec{E} = -\frac{\partial \mu_0 \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t}$$

Take the dot-product of the first equation with  $\vec{H}$  and of the second equation with  $\vec{E}$  to get:

$$(\nabla \times \vec{E}) \cdot \vec{H} = -\frac{1}{2} \frac{\partial \mu_0 \vec{H} \cdot \vec{H}}{\partial t}$$

$$(\nabla \times \vec{H}) \cdot \vec{E} = \vec{J} \cdot \vec{E} + \frac{1}{2} \frac{\partial \epsilon \vec{E} \cdot \vec{E}}{\partial t}$$

Subtract the above two to get:

$$(\nabla \times \vec{E}) \cdot \vec{H} - (\nabla \times \vec{H}) \cdot \vec{E} = -\frac{1}{2} \frac{\partial \mu_0 \vec{H} \cdot \vec{H}}{\partial t} - \vec{J} \cdot \vec{E} - \frac{1}{2} \frac{\partial \epsilon \vec{E} \cdot \vec{E}}{\partial t}$$

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### The Electromagnetic Power-Energy Continuity Equation - II

$$(\nabla \times \vec{E}) \cdot \vec{H} - (\nabla \times \vec{H}) \cdot \vec{E} = -\frac{1}{2} \frac{\partial \mu_0 \vec{H} \cdot \vec{H}}{\partial t} - \vec{J} \cdot \vec{E} - \frac{1}{2} \frac{\partial \epsilon \vec{E} \cdot \vec{E}}{\partial t}$$

Use the vector identity on the left hand side:  $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$

To get:  $-\nabla \cdot (\vec{E} \times \vec{H}) = \frac{\partial}{\partial t} \left( \frac{\mu_0 \vec{H} \cdot \vec{H}}{2} + \frac{\epsilon \vec{E} \cdot \vec{E}}{2} \right) + \vec{J} \cdot \vec{E}$

Define:  $\vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t)$

$$W(\vec{r}, t) = \left( \frac{\mu_0 \vec{H}(\vec{r}, t) \cdot \vec{H}(\vec{r}, t)}{2} + \frac{\epsilon \vec{E}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)}{2} \right)$$

And get:

$$-\nabla \cdot \vec{S}(\vec{r}, t) = \frac{\partial W(\vec{r}, t)}{\partial t} + \vec{J}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)$$

This is what we wanted (other than the last term on the right hand side)

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### The Electromagnetic Power-Energy Continuity Equation - III

Poynting Vector:  $\vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t)$

- $\vec{S}(\vec{r}, t)$  is called the Poynting vector and it is the energy flow per second per unit area
- $\vec{S}(\vec{r}, t)$  points in the direction of power flow
- $\vec{S}(\vec{r}, t)$  has the units of Joules/(m<sup>2</sup>-sec) or Watt/m<sup>2</sup>

Energy Density:  $W(\vec{r}, t) = \left( \frac{\mu_0 \vec{H}(\vec{r}, t) \cdot \vec{H}(\vec{r}, t)}{2} + \frac{\epsilon \vec{E}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)}{2} \right)$

- $W(\vec{r}, t)$  is the energy density of electromagnetic field
- $W(\vec{r}, t)$  has the units of Joules/m<sup>3</sup>
- $W(\vec{r}, t)$  has two parts:
 

$\frac{\mu_0 \vec{H}(\vec{r}, t) \cdot \vec{H}(\vec{r}, t)}{2}$	→	Energy density of the magnetic field
$\frac{\epsilon \vec{E}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)}{2}$	→	Energy density of the electric field

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## The Electromagnetic Power-Energy Continuity Equation - IV

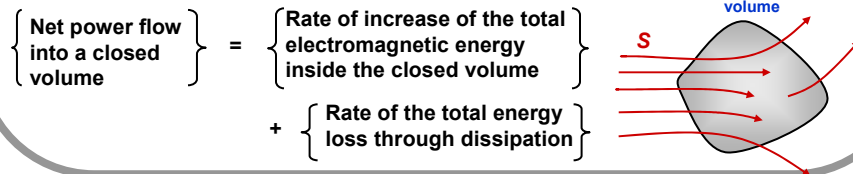
Power dissipation:

$$-\nabla \cdot \vec{S}(\vec{r}, t) = \frac{\partial W(\vec{r}, t)}{\partial t} + \vec{J}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)$$

- The last term in the continuity equation  $J \cdot E$  indicates power dissipation
- Its units are Joules/(m<sup>3</sup>-sec)
- This term represents the energy that is lost from the electromagnetic field

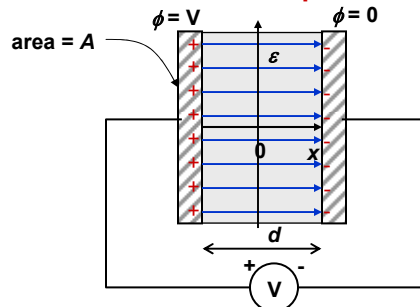
Essentially what is happening is that if the medium is conductive then in the presence of electric fields currents will be produced and the last term represents the usual  $I^2R$  losses in conductors or resistors

**Integral Form:**  $-\oiint \vec{S}(\vec{r}, t) \cdot d\vec{a} = \frac{\partial}{\partial t} \iiint W(\vec{r}, t) dV + \iiint \vec{J}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t) dV$



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### Example: Parallel Plate Capacitor



Recall that:

$$\phi(x) = V \left( \frac{1}{2} - \frac{x}{d} \right)$$

$$E_x(x) = -\frac{\partial \phi(x)}{\partial x} = \frac{V}{d}$$

**Electric field energy density** =  $\frac{\epsilon E_x(x) \cdot E_x(x)}{2} = \frac{\epsilon (V/d)^2}{2}$

**Total electric field energy** =  $\frac{\epsilon E_x(x) \cdot E_x(x)}{2} (Ad) = \frac{\epsilon (V/d)^2}{2} (Ad)$   
 $= \frac{1}{2} CV^2$        $\left\{ C = \frac{\epsilon A}{d} \right.$

**For all (linear) capacitors the total stored electric field energy equals  $CV^2/2$**

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### Example: Parallel Plate Inductor

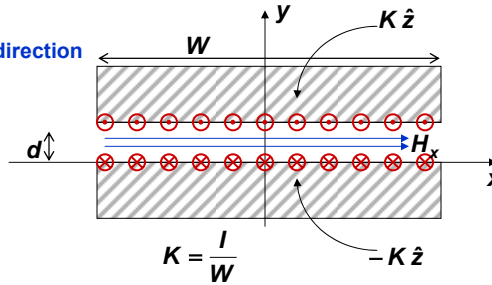
The structure has length  $\ell$  in the z-direction

Recall that:

$$H_x = K = \frac{I}{W}$$

$$\Rightarrow L = \frac{\mu_0 H_x d \ell}{I} = \frac{\mu_0 d \ell}{W}$$

(L is the total inductance)



Magnetic field energy density =  $\frac{\mu_0 H_x \cdot H_x}{2} = \frac{\mu_0}{2} \left( \frac{I}{W} \right)^2$

Total magnetic field energy =  $\frac{\mu_0 H_x \cdot H_x}{2} (W d \ell) = \frac{\mu_0}{2} \left( \frac{I}{W} \right)^2 (W d \ell)$

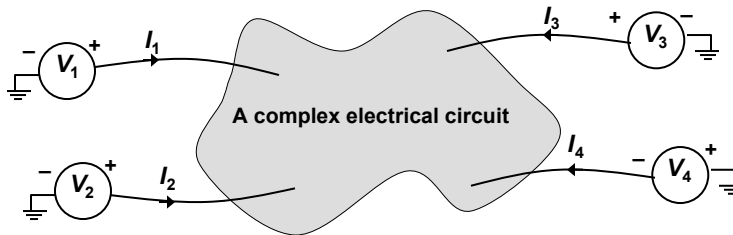
$$= \frac{1}{2} L I^2$$

For all (linear) inductors the total stored magnetic field energy equals  $L I^2 / 2$

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### Power Flow in Electroquasistatics and Circuit Theory - I

Consider a complex electrical circuit shown below:



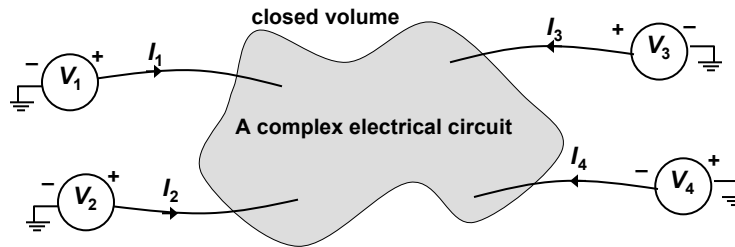
If you wanted to calculate the total electrical power  $P$  going into the circuit you would do the following sum:

$$P = \sum_n V_n I_n = V_1 I_1 + V_2 I_2 + V_3 I_3 - V_4 I_4$$

Where does this formula comes from?

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### Power Flow in Electroquasistatics and Circuit Theory - II



Start from the Poynting Vector:  $\vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t)$

Total power flow into the closed volume =  $P = -\oiint \vec{S} \cdot d\vec{a} = -\oiint (\vec{E} \times \vec{H}) \cdot d\vec{a}$

In electroquasistatics:  $\vec{E} = -\nabla\phi$

In magnetoquasistatics:  $\nabla \times \vec{H} = \vec{J}$

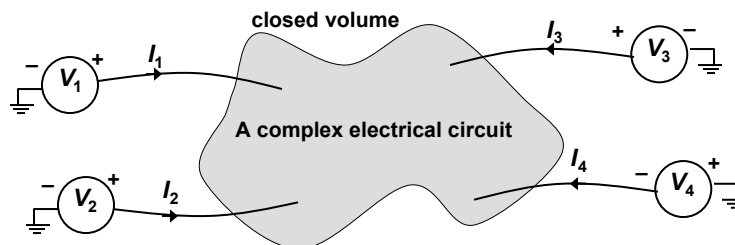
Therefore:  $P = \oiint (\nabla\phi \times \vec{H}) \cdot d\vec{a}$

Use the vector identity:  $\nabla \times (\phi \vec{F}) = (\nabla\phi) \times \vec{F} + \phi (\nabla \times \vec{F})$

To get:  $P = \oiint \nabla \times (\phi \vec{H}) \cdot d\vec{a} - \oiint \phi (\nabla \times \vec{H}) \cdot d\vec{a}$

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### Power Flow in Electroquasistatics and Circuit Theory - III



$P = \oiint \nabla \times (\phi \vec{H}) \cdot d\vec{a} - \oiint \phi (\nabla \times \vec{H}) \cdot d\vec{a}$   $\longrightarrow$   $\left\{ \begin{array}{l} \text{curl of a vector integrated over a} \\ \text{closed surface is always zero} \end{array} \right.$

$\Rightarrow P = -\oiint \phi (\nabla \times \vec{H}) \cdot d\vec{a}$

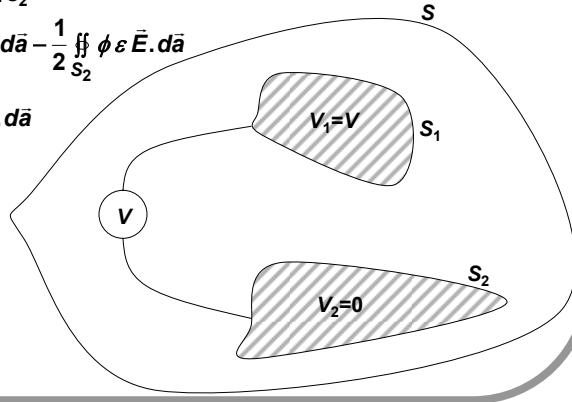
$\Rightarrow P = -\oiint \phi \vec{J} \cdot d\vec{a} = \sum_n V_n I_n$

$\Rightarrow P = \sum_n V_n I_n \longrightarrow$  Just what we wanted to prove

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### Appendix: Energy Stored in Capacitors is $QV/2$

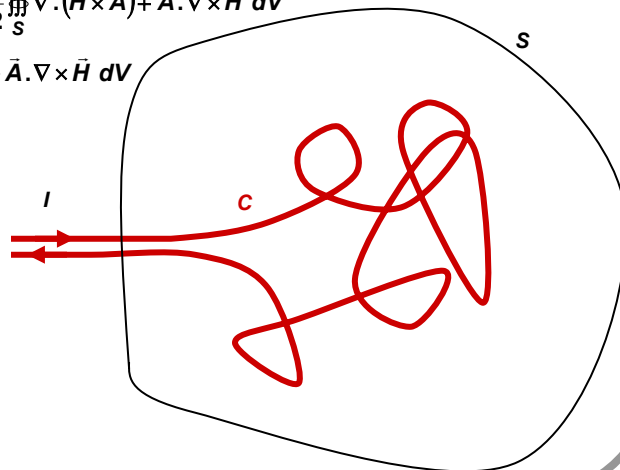
$$\begin{aligned}
 U &= \iiint_{S+S_1+S_2} \frac{\epsilon \vec{E} \cdot \vec{E}}{2} dV \\
 &= -\frac{1}{2} \iiint_{S+S_1+S_2} \epsilon \vec{E} \cdot \nabla \phi dV = -\frac{1}{2} \iiint_{S+S_1+S_2} \nabla \cdot (\phi \epsilon \vec{E}) - \phi \nabla \cdot (\epsilon \vec{E}) dV \\
 &= -\frac{1}{2} \iint_{S+S_1+S_2} \phi \epsilon \vec{E} \cdot d\vec{a} + \frac{1}{2} \iiint_{S+S_1+S_2} \phi \rho dV \\
 &= -\frac{1}{2} \iint_S \phi \epsilon \vec{E} \cdot d\vec{a} - \frac{1}{2} \iint_{S_1} \phi \epsilon \vec{E} \cdot d\vec{a} - \frac{1}{2} \iint_{S_2} \phi \epsilon \vec{E} \cdot d\vec{a} \\
 &= -\frac{1}{2} \iint_{S_1} \phi \epsilon \vec{E} \cdot d\vec{a} - \frac{1}{2} \iint_{S_2} \phi \epsilon \vec{E} \cdot d\vec{a} \\
 &= \frac{1}{2} Q_1 V_1 + \frac{1}{2} Q_2 V_2 \\
 &= \frac{1}{2} QV
 \end{aligned}$$



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### Appendix: Energy Stored in Inductors is $I\lambda/2$

$$\begin{aligned}
 U &= \iiint_S \frac{\mu_0 \vec{H} \cdot \vec{H}}{2} dV \\
 &= \frac{1}{2} \iiint_S \vec{H} \cdot \nabla \times \vec{A} dV = \frac{1}{2} \iiint_S \nabla \cdot (\vec{H} \times \vec{A}) + \vec{A} \cdot \nabla \times \vec{H} dV \\
 &= \frac{1}{2} \iint_S (\vec{H} \times \vec{A}) \cdot d\vec{a} + \frac{1}{2} \iiint_S \vec{A} \cdot \nabla \times \vec{H} dV \\
 &= \frac{1}{2} \iiint_S \vec{A} \cdot \nabla \times \vec{H} dV \\
 &= \frac{1}{2} \iiint_S \vec{A} \cdot \vec{J} dV \\
 &= \frac{1}{2} I \int_C \vec{A} \cdot d\vec{s} \\
 &= \frac{1}{2} I \lambda
 \end{aligned}$$



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