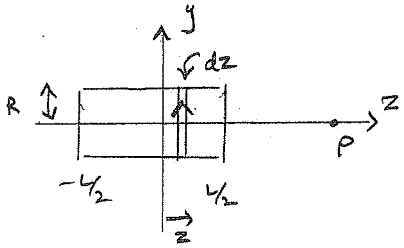
ECE 303 Final Exam Solutions (by Farhan Rana)

Problem 1

a)



(side view)

$\vec{H}$ will only have a z-component (by symmetry) at P.

Using Biot-Savart law, Hz produced by a loop of thickness dz, carrying current Kdz is

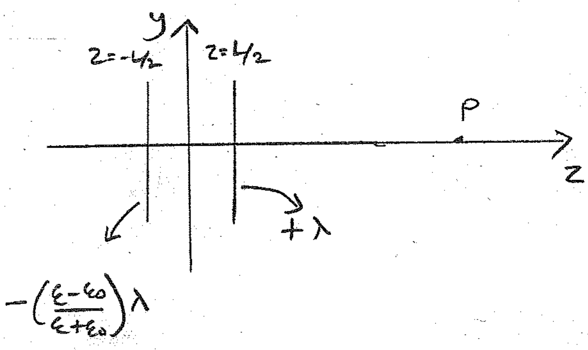
$$= -\frac{Kdz}{4\pi}\frac{2\pi R}{(d-z)^2+R^2}\cdot\frac{R}{\sqrt{(d-z)^2+R^2}}$$

$$= -\frac{K}{2}\frac{R^2\,dz}{\left[(d-z)^2+R^2\right]^{3/2}}$$

$\Rightarrow$ Total Hz at P is:

$$H_z = -\int_{-L/2}^{L/2}\frac{K}{2}\frac{R^2\,dz}{\left[(d-z)^2+R^2\right]^{3/2}}$$

b) From homework #3, a charge Q in front of a dielectric medium of permittivity ε produces an image charge which is of strength $-\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\right)Q$. So the loop of charge density $+\lambda$ will have an image loop of charge density equal to $-\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\right)\lambda$

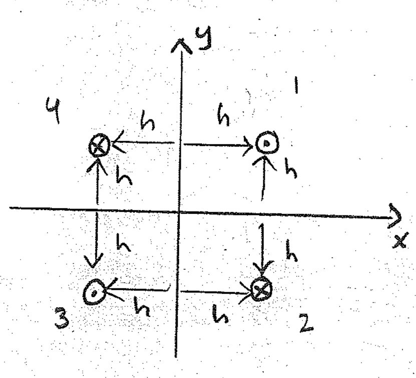


By symmetry, $\vec{E}$ will only have a z-component at P.

$$E_z = \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi a}{\left[(d-L/2)^2+a^2\right]}\cdot\frac{(d-L/2)}{\sqrt{(d-L/2)^2+a^2}}$$

$$-\frac{\lambda}{4\pi\varepsilon_0}\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\right)\frac{2\pi a}{\left[(d+L/2)^2+a^2\right]}\cdot\frac{(d+L/2)}{\sqrt{(d+L/2)^2+a^2}}$$

Problem 2



The image dipoles are as shown.

a) $E_{ff}(\vec{r}) = \{\text{element factor}\}\times\{\text{array factor}\}$ for $x>0$, $y>0$

element factor $= \hat{\theta}\, j\frac{kId_{eff}}{4\pi r}e^{-jkr}\sin\theta$ $\{d_{eff}=\frac{d}{2}$

array factor $= e^{jk\hat{r}\cdot\vec{h}_1} - e^{jk\hat{r}\cdot\vec{h}_2} + e^{jk\hat{r}\cdot\vec{h}_3} - e^{jk\hat{r}\cdot\vec{h}_4}$

$\hat{r}\cdot\vec{h}_1 = h\sin\theta\cos\phi + h\sin\theta\sin\phi$ $\hat{r}\cdot\vec{h}_2 = h\sin\theta\cos\phi - h\sin\theta\sin\phi$

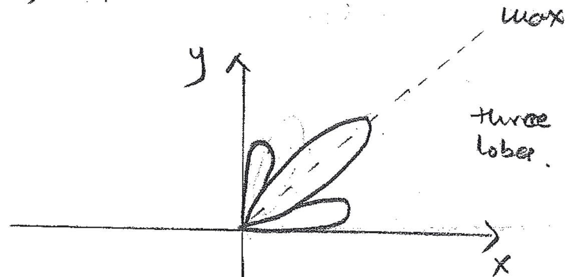
$\hat{r}\cdot\vec{h}_3 = -[h\sin\theta\cos\phi + h\sin\theta\sin\phi]$ $\hat{r}\cdot\vec{h}_3 = -h\sin\theta\cos\phi + h\sin\theta\sin\phi$.

$$\text{array factor} = 2\cos\left[kh\sin\theta(\cos\phi+\sin\phi)\right] - 2\cos\left[kh\sin\theta(\cos\phi-\sin\phi)\right]$$

b) $G(\theta,\phi) = \frac{3}{2}\sin^2\theta\,|\text{array factor}|^2 \Rightarrow G(\theta,\phi)$ will be maximum

when $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{4}$, and $\sqrt{2}kh = \pi$ c) for $h=\lambda$ $p(\theta,\phi)$ is

$\Rightarrow h = \frac{\lambda}{2\sqrt{2}}$ and $G(\theta,\phi)|_{max} = 3\text{x}16/2 = 24$



$h = \frac{\lambda}{2}$

Problem 3

a) $\rho_p = -\nabla\cdot\vec{P} \Rightarrow$ The component of $\vec{P}$ normal to the surface is

$\vec{P} = P_o\hat{z} = P_o(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$ } equal to $P_o\cos\theta$.

$\Rightarrow \sigma_p(\theta) = P_o\cos\theta$.

b) $\Phi_{out}(\vec{r}) = A\left(\frac{R}{r}\right)^2\cos\theta$ $\longrightarrow$ dipole like solution.

$\Phi_{in}(\vec{r}) = B\left(\frac{r}{R}\right)\cos\theta$ $\longrightarrow$ uniform field solution.

c) at $r=R$ $\Phi_{in} = \Phi_{out} \Rightarrow A = B$.

at $r=R$ $E_{r,out} - E_{r,in} = \frac{\sigma_p}{\varepsilon_o} = \frac{P_o\cos\theta}{\varepsilon_o}$

$\frac{2A}{R} + \frac{B}{R} = \frac{P_o}{\varepsilon_o}\cos\theta$

} $A = \frac{P_o R}{3\varepsilon_o} = B$.

Problem 4

a) Need the impedance looking into the transmission line to be Z_A^* for maximum power transfer. $\frac{Z_A^*}{Z_o}$ is shown on the attached Smith chart $\Rightarrow \frac{R_e}{Z_o}$, which is purely real, can have two possible values: Either $\frac{R_s}{Z_0} \cong 2.6$ or $\frac{R_s}{Z_0} \cong 0.38$

$\Rightarrow$ $R_s \cong 260\ \Omega$ or $R_s \cong 38\ \Omega$

b) From Smith chart, when $R_s \cong 38\ \Omega$, $L = .09\lambda$, and

when $R_s \cong 260\ \Omega$, $L = 0.34\lambda$.

c) When R_s and L are as in part (a) + (b), the antenna is also matched to the resistor R_s. So the power delivered to the antenna P is $\frac{|V_s|^2}{8R_s}$ and therefore $P_{rad} = \eta_{rad} \frac{|V_s|^2}{8R_s}$

Problem 5:

a). $\vec{E}$ field is in entirely in the $\hat{y}$-direction so the wave sees the medium as having permittivity $\varepsilon = (1-5j)\varepsilon_0$. Phase matching at the boundary $\Rightarrow k_{ix} = k_{tx} \Rightarrow k_{tx} = \frac{\omega}{c}\sin\theta_i$ $\left\{ \theta_i = \frac{\pi}{4} = 45° \right\}$

$$k_{tz} = \sqrt{\frac{\omega^2}{c^2}(1-5j) - k_{tx}^2} = \frac{\omega}{c}\sqrt{(1-5j) - \frac{1}{2}} = \frac{\omega}{c}\frac{1}{\sqrt{2}}\sqrt{1-10j}$$

$$\theta_t = \tan^{-1}\left\{\frac{k_{tx}}{Re\{k_{tz}\}}\right\} = \tan^{-1}\left\{\frac{1/\sqrt{2}}{\frac{1}{\sqrt{2}} Re\{\sqrt{1-10j}\}}\right\} = \tan^{-1}\left\{\frac{1}{Re\sqrt{1-10j}}\right\} \cong 23°$$

b) At cut-off the angle θ_i for any mode is the critical angle corresponding to the interface that has the larger critical angle. In this case, $\theta_i = \theta_c = \sin^{-1}\left\{\frac{2.5}{3.0}\right\} \cong 56°$

Problem 4: parts (a), (b) and (c)

Name: ____