School of Electrical and Computer Engineering, Cornell University

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ECE 303: Electromagnetic Fields and Waves

Fall 2007

Final Exam **December 12, 2007**

INSTRUCTIONS:

• Only work done on the blue exam booklets will be graded – do not attach your own sheets to the exam booklets under any circumstances

• To get partial credit you must show all the relevant work

• Correct answers with wrong reasoning will not get points

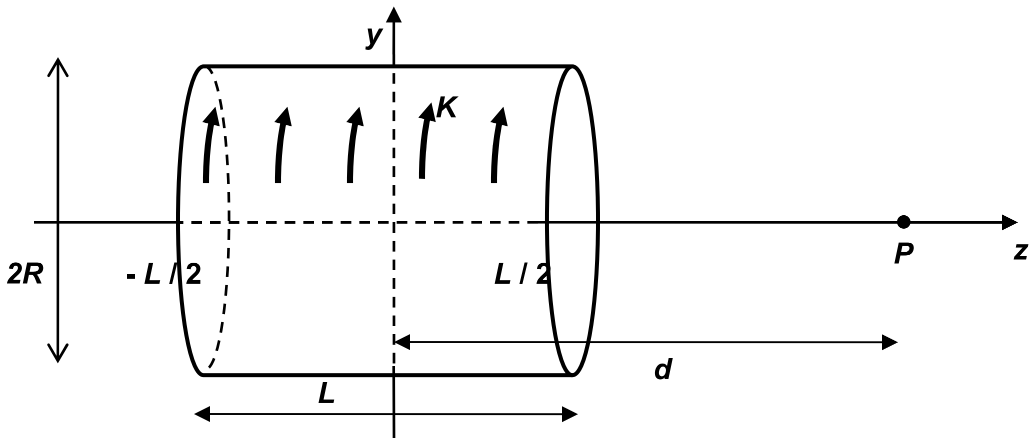
• All questions/parts do not carry equal points

• All questions do not have the same level of difficulty

DO NOT WRITE IN THIS SPACE

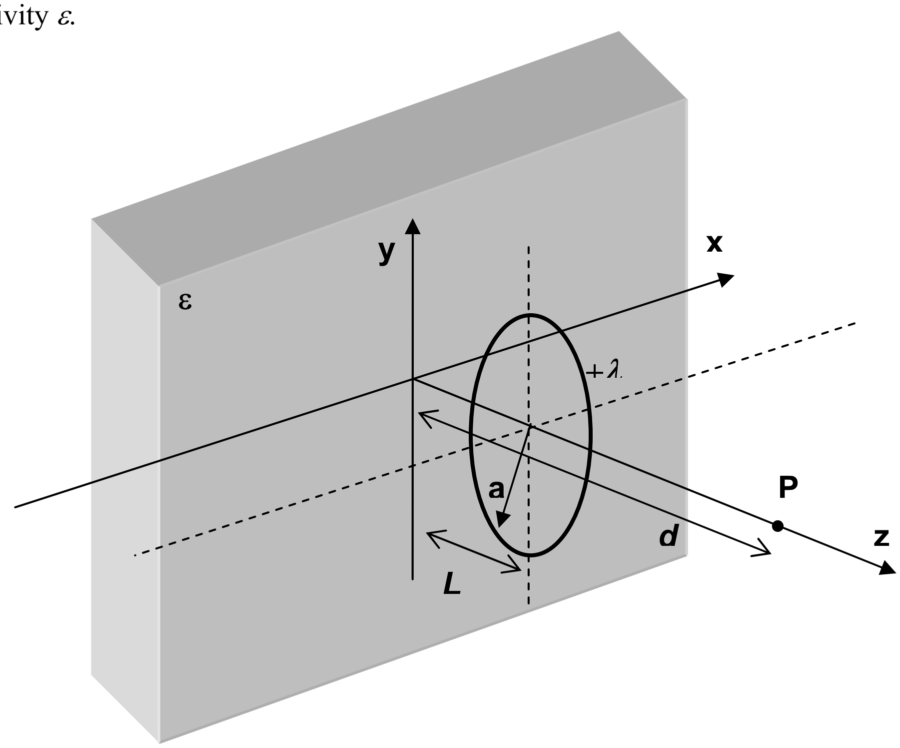
Problem 1 (20 points)

a) Consider a thin cylindrical shell of length L and radius R located with its center at the origin and oriented along the z-direction, as shown in the figure below. The surface of the cylinder carries a surface current density (units: Amps/m) given by $\vec{K} = -K\hat{\phi}$.



Find the exact magnetic field vector (magnitude and direction) at the point P which is located at a distance d from the origin along the z-direction, as shown in the figure above. You may leave your answer in the form of a definite integral.

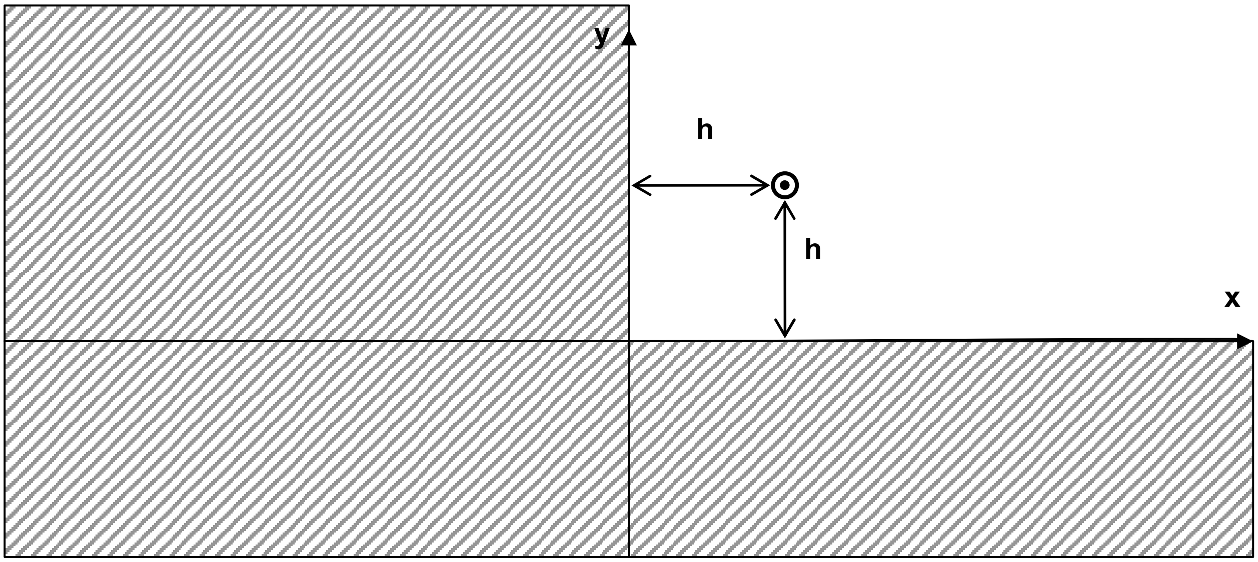
b) Consider a circular loop of line charge of radius a and carrying a charge density $+\lambda$ per unit length, as shown below. The loop is positioned at a distance d along the z-axis from the x-y plane and is oriented parallel to the x-y plane. The infinite space for all negative values of the coordinate z is occupied by a medium of permittivity ε.



Find the exact electric field vector (magnitude and direction) at the point P which is located at a distance d from the origin along the z-direction, as shown in the figure above. You may leave your answer in the form of a definite integral.

Problem 2 (20 points)

Consider a short dipole of physical length d, pointing in the z-direction, carrying a current phasor I, and located at $(h,h,0)$ in the (x,y,z) coordinate system, as shown in the figure below. The space for which $x<0$ or $y<0$ is filled with a perfect metal.



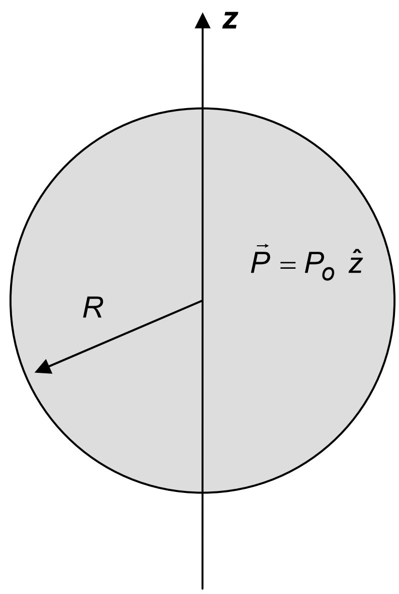
a) Find the expression for the far-field electric field vector $\vec{E}_{ff}(\vec{r})$ of the radiation and explain your result.

b) Find an exact expression for the gain $G(\theta,\phi)$ of the antenna. What value of h (in terms of the wavelength λ) will give the maximum value for the Gain and in which directions (θ,ϕ) does this maximum value of Gain occur?

c) Sketch the radiation pattern $p(\theta,\phi)$ in the **x-y plane** assuming that the distance h equals λ.

Problem 3 (20 points)

Consider a ferroelectric material in the form of a sphere of radius R (recall from homework that in ferroelectric materials the atoms/molecules are polarized even in the absence of any external applied field). The material has a has a built-in, non-zero, fixed and uniform polarization vector $\vec{P}$ that is independent of any external field and is given by $\vec{P} = P_o \ \hat{z}$. The permittivity everywhere is ε_o,



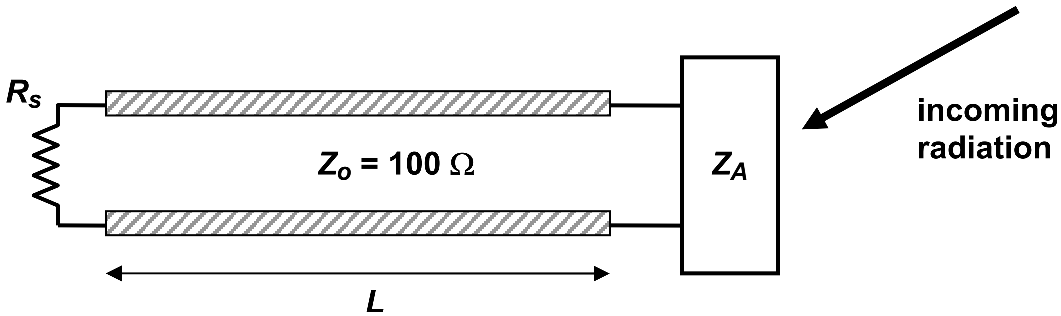
a) Find the surface charge density σ_p due to the paired charges at the surface of the sphere. Explain your answer.

b) Write trial solutions for the potentials, $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$, in the two regions $0 \le r \le R$ and $R \le r \le \infty$, respectively.

c) Using all the boundary conditions at your disposal, find the potentials $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$.

Problem 4 (20 points)

Consider an antenna with an effective area given by $A(\theta,\phi)$, radiative efficiency given by η_{rad} , and impedance given by $Z_A = 100 + j100\ \Omega$. The antenna is connected to a transmission line of impedance $Z_o = 100\ \Omega$ which in turn is connected to a resistor R_S , as shown below. The wavelength of the incoming radiation is λ. If you need to use a Smith chart for parts (a), (b) and (c) of this problem you will find one attached at the end of this exam.

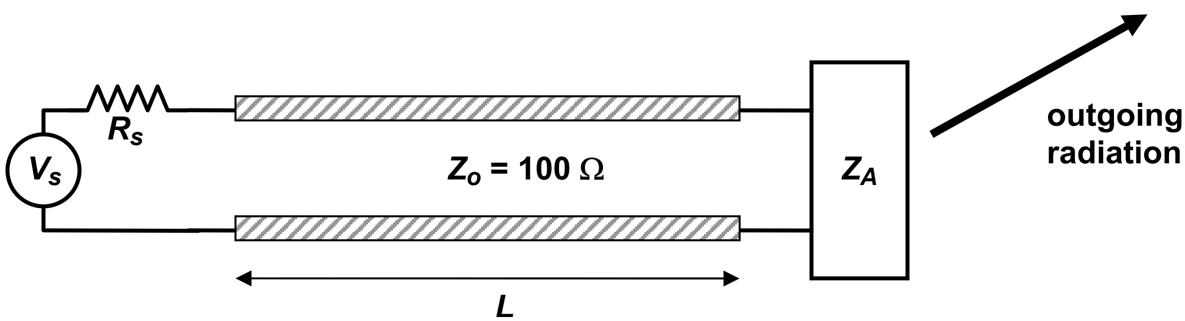


Assume first that the antenna is used as a receiver. You need to choose the length L of the transmission line and the value of the resistor R_S to deliver the maximum power from the antenna to the resistor R_S .

a) What value of the resistor R_S will result in the maximum power transfer from the antenna to the resistor R_S . Give a numerical value. Explain your answer.

b) For the value of the resistor R_S chosen in part (a) indicate the length L of the transmission line (in terms of λ) needed to enable the maximum transfer of power from the antenna to the resistor R_S . Explain your answer.

Now suppose the antenna receiver is used as a transmitter and is driven by a voltage source whose phasor is V_S , as shown below.



c) For the value of the resistor R_S chosen in part (a) and the corresponding length L of the transmission line chosen in part (b), what is the time-averaged power P_{rad} radiated by the antenna? Give your answer in terms of the voltage source phasor V_S .

Problem 5 (20 points)

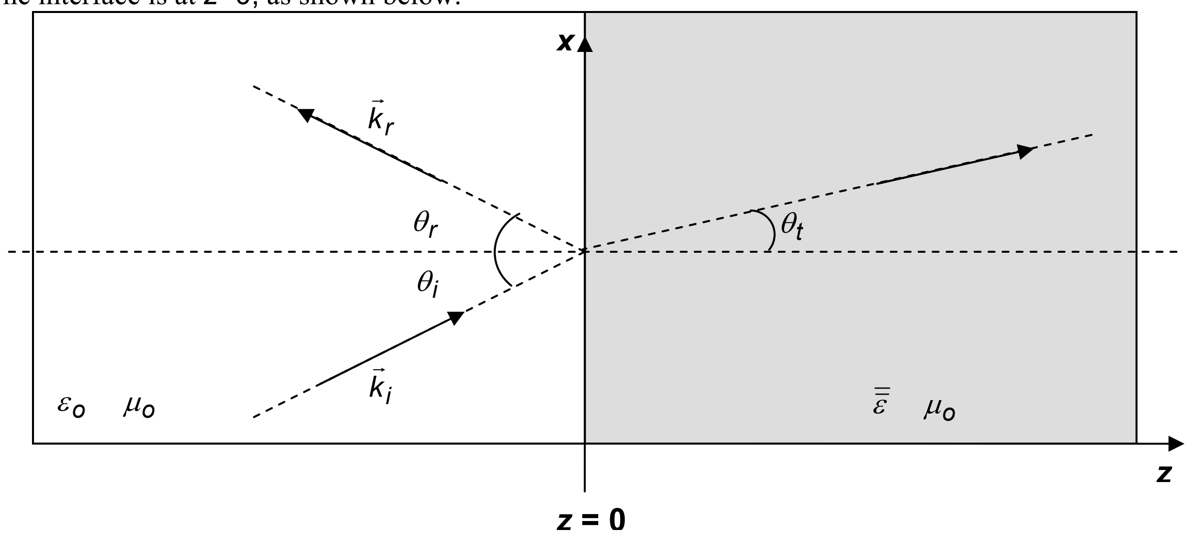
a) Consider a plane wave of frequency ω and given by the expression:

$$\vec{H}(\vec{r}) = (\hat{x} H_o - \hat{z} H_o)\, e^{-j\,\vec{k}_i \cdot \vec{r}}$$

incident from free space onto an interface with a medium whose permittivity is given by the tensor:

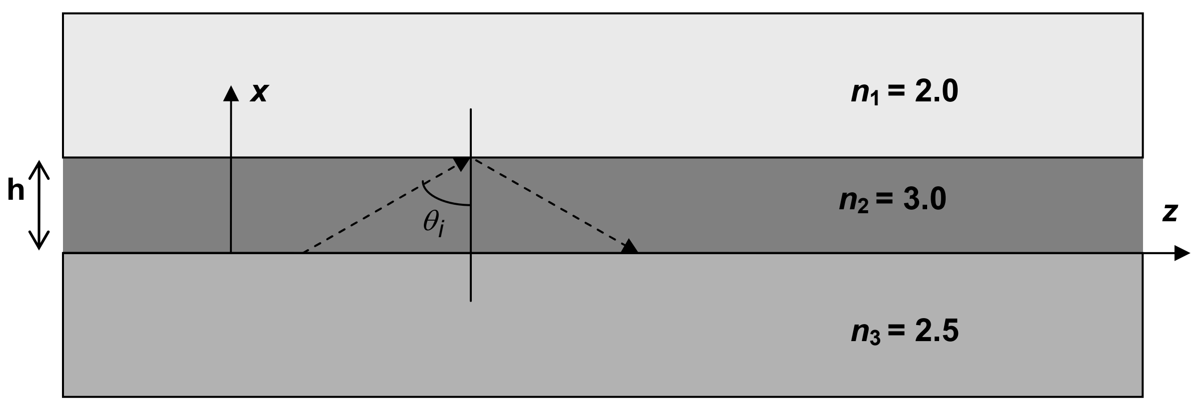
$$\bar{\bar{\varepsilon}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & (1-5j) & 0 \\ 0 & 0 & 4 \end{bmatrix} \varepsilon_o$$

The interface is at $z=0$, as shown below.



What is angle of transmission θ_t ? Show all your work. Give a numerical value as an answer.

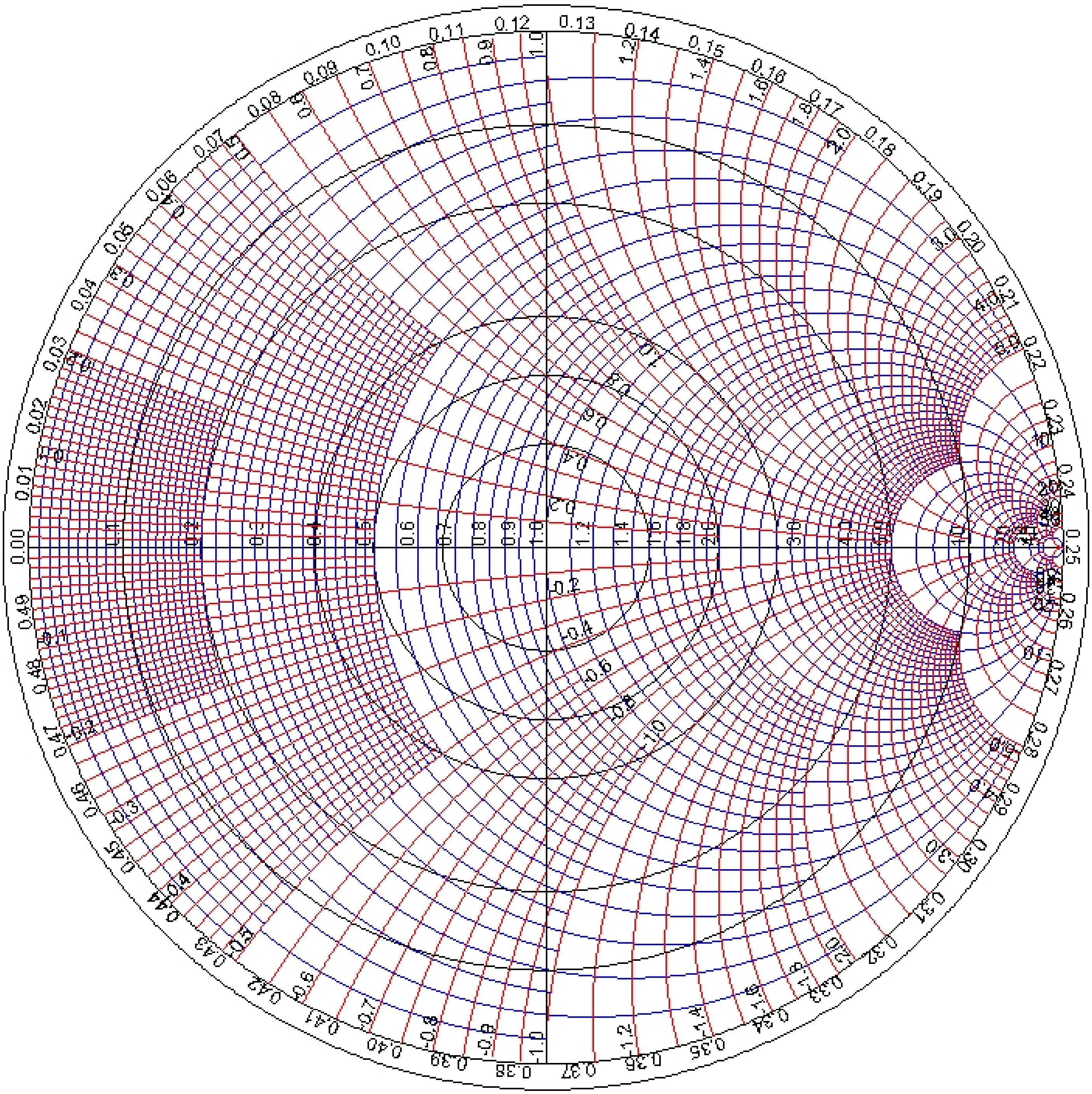
b) Consider a dielectric slab waveguide, as shown in the figure below. The thickness h of the core of the waveguide is 1.0 micrometer.



Find the angle of incidence θ_i of the guided TM_2 mode wave at the interface between the core of refractive index 3.0 and the cladding of refractive index 2.0 at a frequency close to the cut-off frequency of the TM_2 mode.

Problem 4: parts (a), (b) and (c)

Name: ____



ECE 303: Electromagnetic Fields and Waves
The Complete Smith Chart

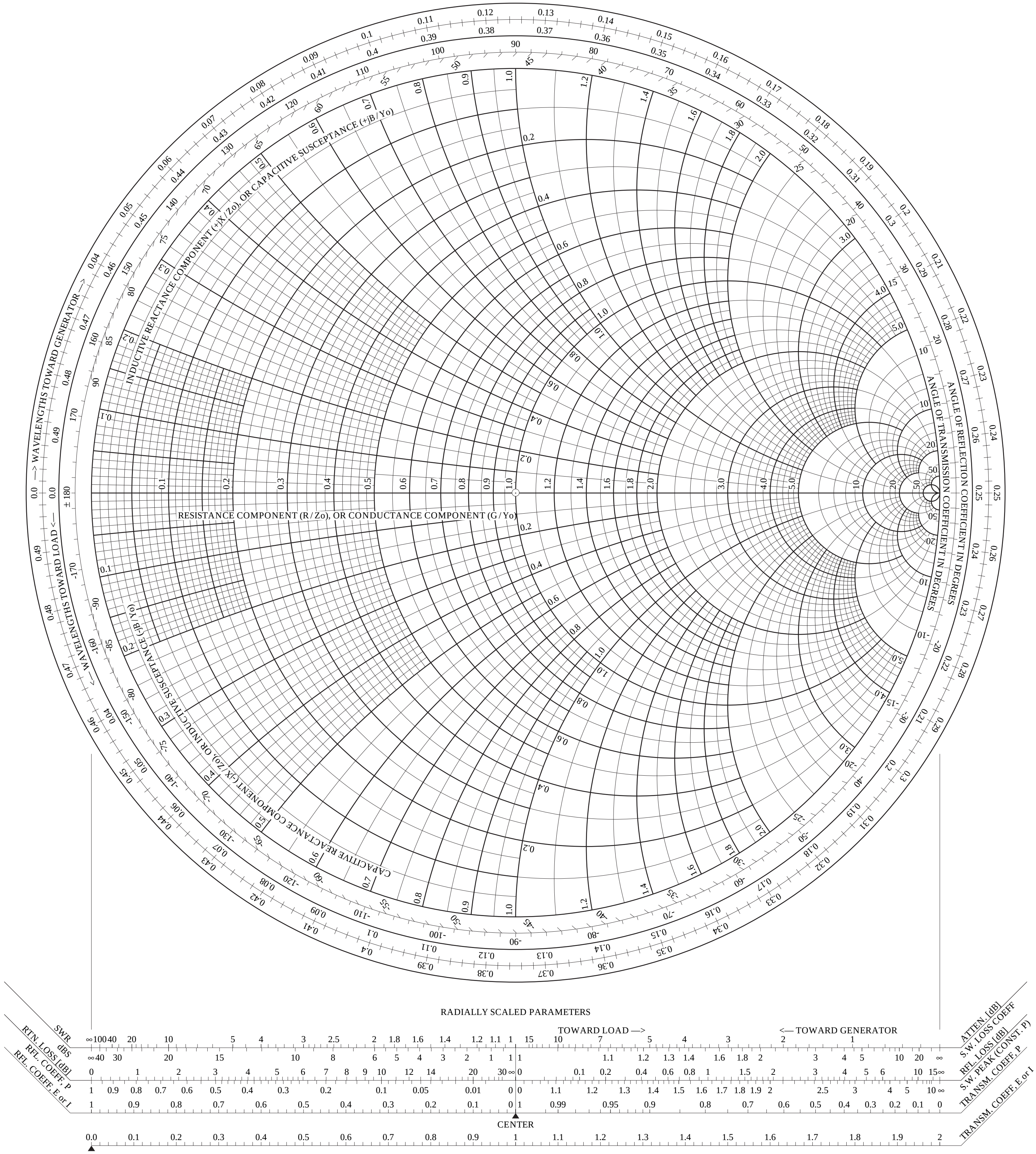


Table of Solutions of Laplace's Equation

Spherical Coordinate System	Cylindrical Coordinate System
$\phi(\vec{r}) = A$ Constant potential	$\phi(\vec{r}) = A$ Constant potential
$\phi(\vec{r}) = \frac{A}{r}$ Spherically symmetric potential	$\phi(\vec{r}) = A\ln(r)$ Cylindrically symmetric potential
$\phi(\vec{r}) = A\, r\cos(\theta)$ Potential for uniform z-directed E-Field	$\phi(\vec{r}) = A\, r\cos(\phi)$ Potential for uniform x-directed E-Field
	$\phi(\vec{r}) = A\, r\sin(\phi)$ Potential for uniform y-directed E-Field
$\phi(\vec{r}) = A\frac{\cos(\theta)}{r^2}$ Potential for point-charge-dipole-like solution oriented along the z-axis	$\phi(\vec{r}) = A\frac{\cos(\phi)}{r}$ Potential for line-charge-dipole-like solution oriented along the x-axis
	$\phi(\vec{r}) = A\frac{\sin(\phi)}{r}$ Potential for line-charge-dipole-like solution oriented along the y-axis

Maxwell's Equations

Differential Form

$$\nabla \cdot \vec{D} = \rho_u$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J}_u + \frac{\partial \vec{D}}{\partial t}$$

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Integral Form

$$\oint\!\!\oint \vec{D} \cdot d\vec{a} = \iiint \rho_u \; dV$$

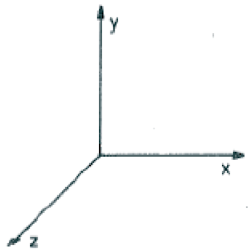
$$\oint\!\!\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial \iint \vec{B} \cdot d\vec{a}}{\partial t}$$

$$\oint \vec{H} \cdot d\vec{s} = \iint \vec{J}_u \cdot d\vec{a} + \frac{\partial \iint \vec{D} \cdot d\vec{a}}{\partial t}$$

Table I. Differential Operators in Cartesian, Cylindrical and Spherical Coordinates

CARTESIAN



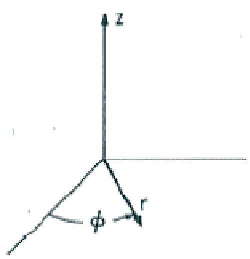
$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{i}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{i}_z$$

$$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{i}_x + \frac{\partial U}{\partial y}\boldsymbol{i}_y + \frac{\partial U}{\partial z}\boldsymbol{i}_z$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

CYLINDRICAL



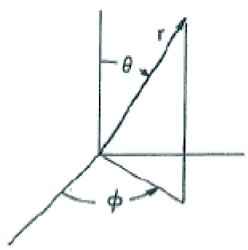
$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \boldsymbol{i}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \boldsymbol{i}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \boldsymbol{i}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla U = \boldsymbol{i}_r\frac{\partial U}{\partial r} + \boldsymbol{i}_\phi\frac{1}{r}\frac{\partial U}{\partial \phi} + \boldsymbol{i}_z\frac{\partial U}{\partial z}$$

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

SPHERICAL



$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \boldsymbol{i}_r\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}\right]$$

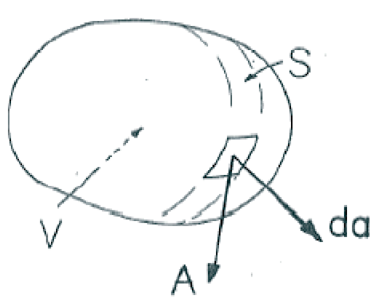
$$+ \boldsymbol{i}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \boldsymbol{i}_\phi\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla U = \boldsymbol{i}_r\frac{\partial U}{\partial r} + \boldsymbol{i}_\theta\frac{1}{r}\frac{\partial U}{\partial \theta} + \boldsymbol{i}_\phi\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

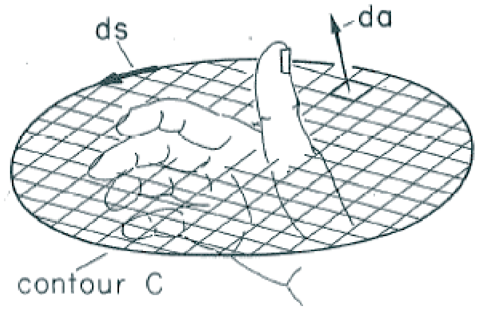
Table II. Integral Theorems

GAUSS'



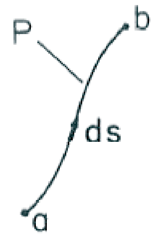
$$\int_V \nabla \cdot \mathbf{A}\, dV = \oint_S \mathbf{A} \cdot d\mathbf{a}$$

STOKES'



$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{s}$$

GRADIENT



$$\int_a^b \nabla U \cdot d\mathbf{s} = U_b - U_a$$

Table III. Vector Identities

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A}$$

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

$$\nabla \times \nabla\phi = 0$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

Table IV. Basic Constants

Permittivity of free space	$\epsilon_0 = 8.854 \times 10^{-12} F/m$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7} H/m$
Speed of light in free space	$c = 1/\sqrt{\mu\epsilon} = 2.9979 \times 10^8 m/s$
Impedance of free space	$\sqrt{\mu_0/\epsilon_0} = 376.7 V/A$
Electron charge	$e = 1.602 \times 10^{-19} C$
Electron mass	$m = 9.11 \times 10^{-31} kg$
Proton mass	$m = 1.67 \times 10^{-27} kg$
Acceleration of gravity	$g = 9.807 m/s^2$