School of Electrical and Computer Engineering, Cornell University

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ECE 303: Electromagnetic Fields and Waves

Fall 2007

Exam 1 **September 25, 2007**

INSTRUCTIONS:

- **Every problem must be done in a separate blue booklet – so you must have 3 separate blue booklets before starting the exam**
- **Only work done on the blue exam booklets will be graded – do not attach your own sheets to the exam booklets under any circumstances**
- **To get partial credit you must show all the relevant work**
- **Correct answers with wrong reasoning will not get points**
- **All questions do not carry equal points**
- **All questions do not have the same level of difficulty**

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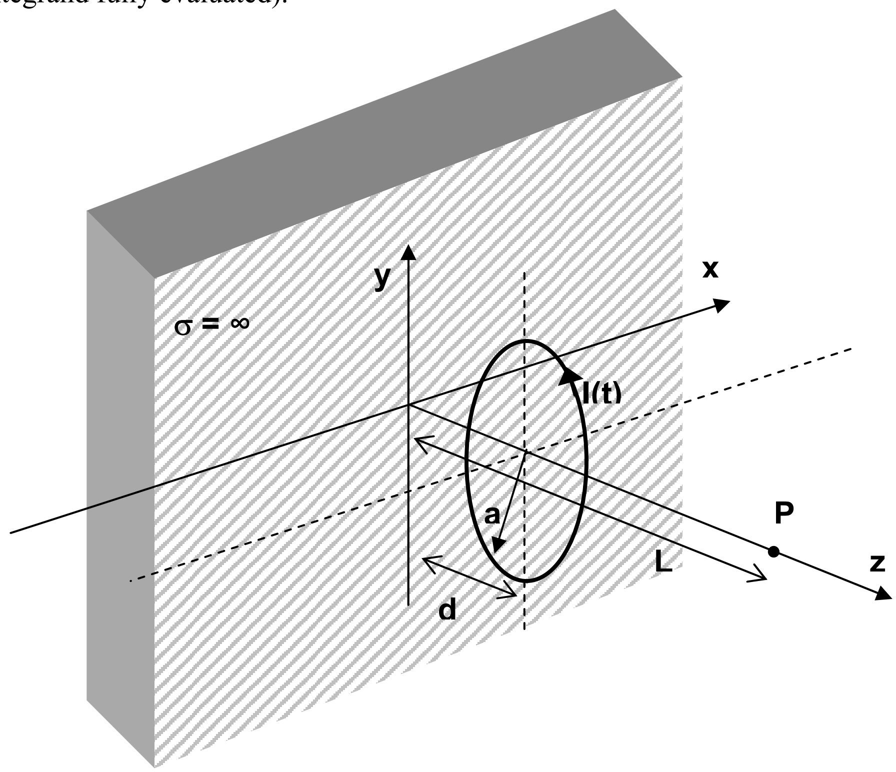
Problem 1 (30 points)

a) Consider a circular loop of current of radius a and carrying a time-dependent current $I(t)$ as shown below. The loop is positioned at a distance d along the z-axis from the x-y plane and is oriented parallel to the x-y plane. The infinite space for all negative values of the coordinate z is occupied by a perfect metal. You can leave your answers to parts (b) and (c) in an integral form (with proper integration limits indicated and the integrand fully evaluated).



a) Find the location and orientation of the image current and specify the magnitude and direction of flow of the image current.

b) Find the z-component of the time-dependent H-field at the location **P** at a distance L along the z-axis.

c) Find the x-component of the time-dependent H-field at the location **P** at a distance L along the z-axis.

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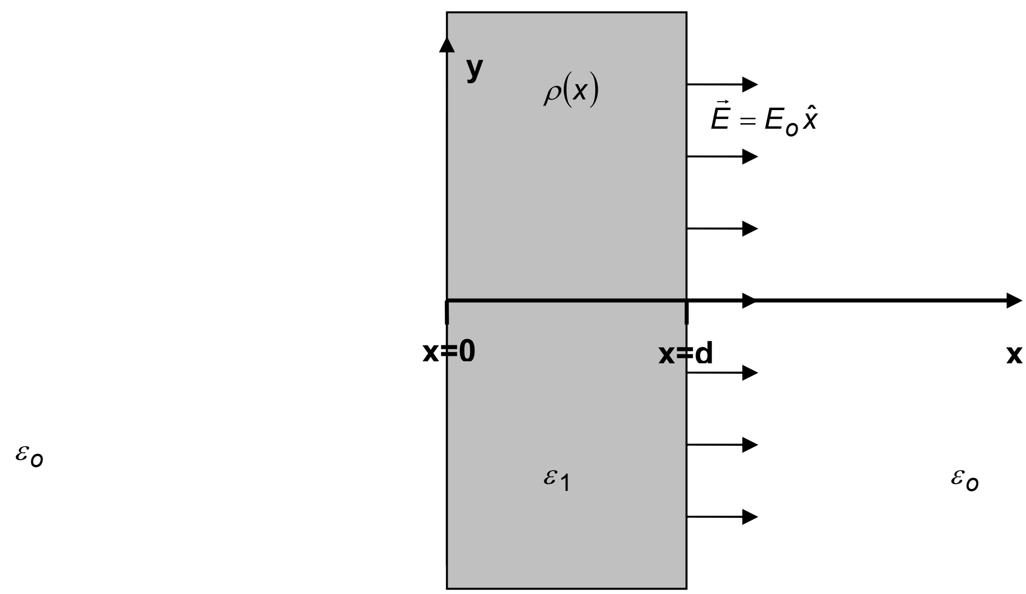
Problem 2 (30 points)

Consider a thin dielectric slab of dielectric constant ε_1 and thickness d. The slab is infinite in the *y* and *z* directions. The slab has a **fixed** position dependent volume charge density (due to unpaired charges) given by $\rho(x)$(units: C/m^3), where:

$$\rho(x) = \rho_o \left(\frac{x}{d}\right)^2$$

It is known that the electric field just to the right of the slab in the air region is given by $\vec{E} = E_o \hat{x}$, as shown below.



a) Find the electric field vector $\vec{E}(x)$ (magnitude and direction) for $0 < x < d$.

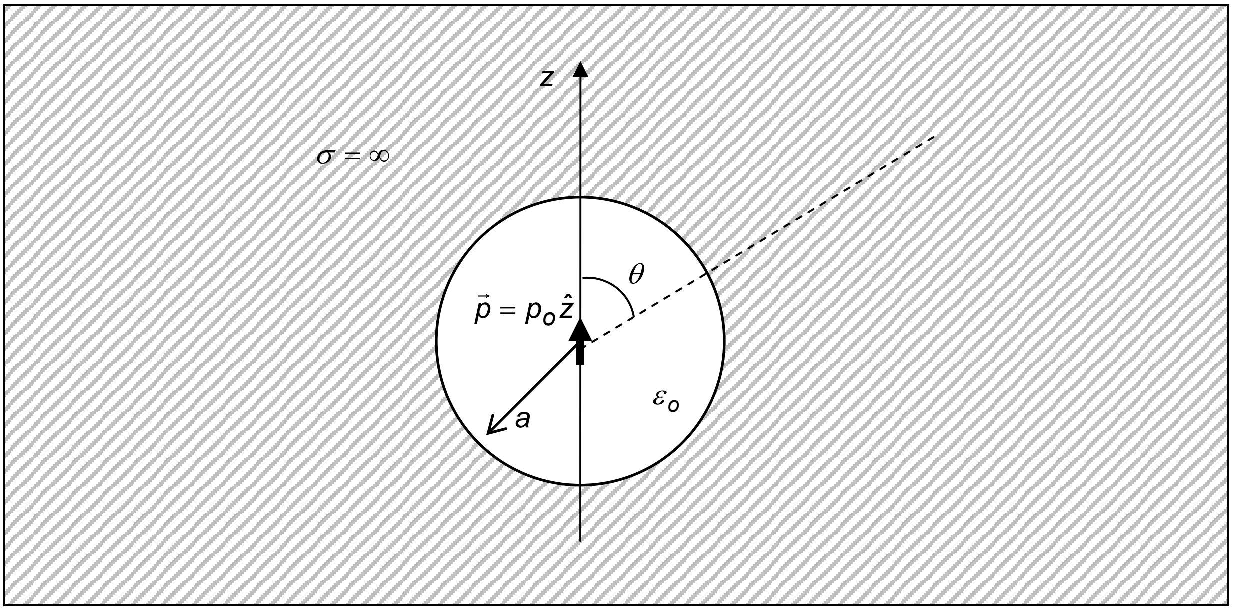
b) Find the electric field vector $\vec{E}(x)$ (magnitude and direction) for $x < 0$.

c) Find the surface charge densities (magnitude and sign) due to paired charges at the air-dielectric interfaces at $x = 0$ and at $x = d$.

DO NOT WRITE IN THIS SPACE

Problem 3 (40 Points)

Consider an infinite **perfect-metal** block. Inside the metal block there is a spherical cavity of radius a. Inside the cavity the dielectric constant everywhere is ε_o. At the center of this spherical cavity there is a very small electric charge dipole sitting, as shown in the figure below. The dipole moment of the dipole is given by, $\vec{p} = p_o \hat{z}$.



a) Find trial solutions, $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$, for the potentials inside and outside the spherical cavity, respectively.

b) Write down all the boundary conditions relevant to solving for the potentials, $\phi_{in}(\vec{r})$ and $\phi_{out}(\vec{r})$ and find all the unknown constants in your solutions in part (a) above by using these boundary conditions.

c) Find the induced surface charge density σ on the inside surface of the spherical cavity.

d) Sketch the E-field lines inside and outside the sphere (points awarded will depend on the quality of the sketch).

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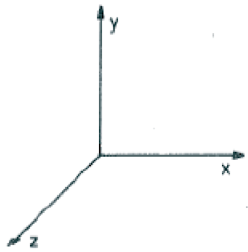
Solutions of Laplace's Equation for ECE303

Spherical Coordinate System	Cylindrical Coordinate System
$\phi(\vec{r}) = A$ Constant potential	$\phi(\vec{r}) = A$ Constant potential
$\phi(\vec{r}) = \frac{A}{r}$ Spherically symmetric potential	$\phi(\vec{r}) = A\ln(r)$ Cylindrically symmetric potential
$\phi(\vec{r}) = A\, r\cos(\theta)$ Potential for uniform z-directed E-Field	$\phi(\vec{r}) = A\, r\cos(\phi)$ Potential for uniform x-directed E-Field
	$\phi(\vec{r}) = A\, r\sin(\phi)$ Potential for uniform y-directed E-Field
$\phi(\vec{r}) = A\frac{\cos(\theta)}{r^2}$ Potential for point-charge-dipole-like solution oriented along the z-axis	$\phi(\vec{r}) = A\frac{\cos(\phi)}{r}$ Potential for line-charge-dipole-like solution oriented along the x-axis
	$\phi(\vec{r}) = A\frac{\sin(\phi)}{r}$ Potential for line-charge-dipole-like solution oriented along the y-axis

Table I. Differential Operators in Cartesian, Cylindrical and Spherical Coordinates

CARTESIAN



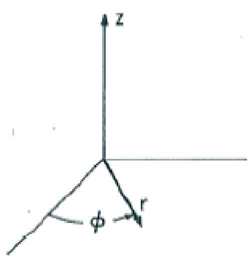
$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{i}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{i}_z$$

$$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{i}_x + \frac{\partial U}{\partial y}\boldsymbol{i}_y + \frac{\partial U}{\partial z}\boldsymbol{i}_z$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

CYLINDRICAL



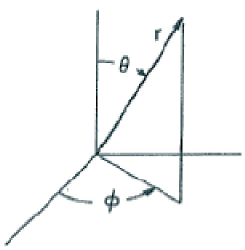
$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \boldsymbol{i}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \boldsymbol{i}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \boldsymbol{i}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla U = \boldsymbol{i}_r\frac{\partial U}{\partial r} + \boldsymbol{i}_\phi\frac{1}{r}\frac{\partial U}{\partial \phi} + \boldsymbol{i}_z\frac{\partial U}{\partial z}$$

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

SPHERICAL



$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

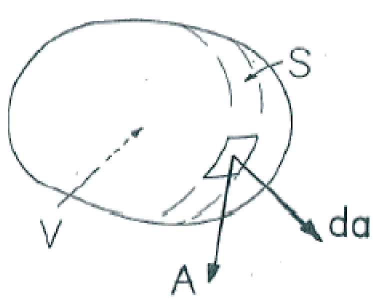
$$\nabla \times \mathbf{A} = \boldsymbol{i}_r\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}\right] + \boldsymbol{i}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \boldsymbol{i}_\phi\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla U = \boldsymbol{i}_r\frac{\partial U}{\partial r} + \boldsymbol{i}_\theta\frac{1}{r}\frac{\partial U}{\partial \theta} + \boldsymbol{i}_\phi\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

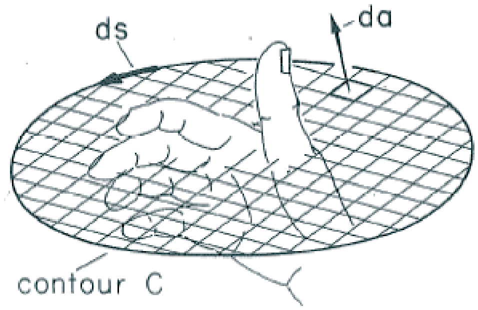
Table II. Integral Theorems

GAUSS'



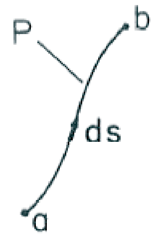
$$\int_V \nabla \cdot \mathbf{A}\,dV = \oint_S \mathbf{A} \cdot d\mathbf{a}$$

STOKES'



$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot ds$$

GRADIENT



$$\int_a^b \nabla U \cdot ds = U_b - U_a$$

Table III. Vector Identities

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A}$$

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

$$\nabla \times \nabla\phi = 0$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

Table IV. Basic Constants

Permittivity of free space	$\epsilon_0 = 8.854 \times 10^{-12} F/m$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7} H/m$
Speed of light in free space	$c = 1/\sqrt{\mu\epsilon} = 2.9979 \times 10^8 m/s$
Impedance of free space	$\sqrt{\mu_0/\epsilon_0} = 376.7 V/A$
Electron charge	$e = 1.602 \times 10^{-19} C$
Electron mass	$m = 9.11 \times 10^{-31} kg$
Proton mass	$m = 1.67 \times 10^{-27} kg$
Acceleration of gravity	$g = 9.807 m/s^2$